LARGE MATCHING MARKETS AS TWO-SIDED DEMAND SYSTEMS

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Abstract. This paper studies two-sided matching markets with non-transferable utility when the number of market participants grows large. We consider a model in which each agent has a random preference ordering over individual potential matching partners, and agents’ types are only partially observed by the econometrician. We show that in a large market, the inclusive value is a sufficient statistic for an agent’s endogenous choice set with respect to the probability of being matched to a spouse of a given observable type. Furthermore, while the number of pairwise stable matchings for a typical realization of random utilities grows at a fast rate as the number of market participants increases, the inclusive values resulting from any stable matching converge to a unique deterministic limit. We can therefore characterize the limiting distribution of the matching market as the unique solution to a fixed point condition on the inclusive values. Finally we analyze identification and estimation of payoff parameters from the asymptotic distribution of observable characteristics at the level of pairs resulting from a stable matching.

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We consider identification and estimation of preference parameters in two-sided matching markets, where the researcher does not observe agents’ preference rankings over matching partners, but only certain attributes or characteristics of the individuals involved in a transaction. Our setup assumes that the market outcome is a pairwise stable matching with non-transferable utilities (NTU), where each agent has a strict preference ordering over individuals on the opposite side of the market, and agents’ types are only partially observed.

In contrast to markets for a homogenous good, matching models describe markets in which agents have preferences regarding the identity of the party they transact or associate with. While we use the language of the classical stable marriage problem to describe our results, there are many other relevant examples for matching markets without transfers between agents, or transfers that are not negotiated individually at the level of the matched pair. This includes mechanisms for assigning students to schools or colleges, of interns to

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hospitals, patients and doctors, or contracts between workers and employers if wages or salaries are determined by centralized bargaining or the government (see e.g. Royal Swedish Academy of Sciences (2012) for a description of several practical applications of matching models). Markets of this type were first analyzed by Gale and Shapley (1962), and existence and properties of pairwise stable matchings are now well understood from a theoretical perspective (see Roth and Sotomayor (1990) for a summary of results). However, estimation of preference parameters from these models remains challenging due to the large number of distinct stable matchings and the interdependence between individuals’ preferences and matching opportunities.

We derive a highly tractable asymptotic approximation to the distribution of matched observable characteristics resulting from pairwise stable matchings when the number of agents in the market is large. We obtain a unique limit for the type-specific matching frequencies that depends on agents’ preferences only through a joint surplus measure at the match level and the inclusive value for the set of matching partners available to the agent. We show that the joint surplus as a function of observable characteristics is nonparametrically identified from the asymptotic distribution. Without further assumptions, it is not possible to identify preferences on the male and female side of the market separately. For estimation of structural parameters from the limiting distribution using likelihood methods, the (endogenously determined) inclusive value functions can be treated as auxiliary parameters which solve the theoretical equilibrium conditions. We also extend the main model to the case in which individuals may only be aware of a random subset of potential matching partners.

Understanding the structure of matching markets without transfers is in itself of economic interest, and it is instructive to compare our findings to known properties of competitive markets with prices and matching markets with transfers. We find that, paralleling our intuitions for competitive markets, pairwise stability generates an essentially unique observable market outcome, where relative transaction (matching) frequencies are a function of a surplus measure at the level of a matched pair. In the absence of side payments or prices, the matching market clears through the relative abundance or scarcity of available matching partners of a certain type rather than explicit transfers. Furthermore, the pseudo-surplus measure captures only stochastic substitution patterns between characteristics of matching partners rather than trade-offs at the individual level, and is not maximized by the market outcome. In particular, the scale of the male and female contributions to pseudo-surplus reflect the importance of systematic utility relative to idiosyncratic taste shifters in either side’s random preferences rather than a common numéraire.

We characterize the stable matching as a result of every agent choosing from the set of potential spouses that are available to her or him. In particular, a potential spouse is available to an agent if and only if s/he prefers that agent to his/her current match. Hence,
the pairwise stability conditions correspond to a discrete choice problem with latent choice sets that are not observed by the researcher and endogenous outcomes of the model. Without further structure, this leads to enormous difficulties for identification analysis or estimation. We show that many commonly used random utility models in fact exhibit independence of irrelevant alternatives (IIA) as a limiting property when the set of choice alternatives is large. Specifically, conditional choice probabilities converge to their Logit analog for a much broader class of random utility model. This leads to a substantial simplification of our analysis because in that limit an individual’s matching opportunities can be summarized by the inclusive value, which a scalar sufficient statistic with respect to the conditional choice probabilities under the Logit model.

We then show that in the limit, the equilibrium inclusive values are a deterministic function of an individual’s observable attributes alone. Furthermore, the inclusive value functions generated by a stable matching are shown to be uniquely determined by the marginal distributions of men’s and women’s observable characteristics. This result does not imply uniqueness of the stable matching in the limit - in fact it is known that the typical number of stable matchings increases at an exponential rate in the number of participants. Rather, the increasing number of distinct matchings are in the limit indistinguishable from the researcher’s perspective since they all result in the same limiting joint distribution of matched observable characteristics.

While there are many possible ways of embedding an $n$-agent economy into a sequence of markets, our main objective in this paper is to construct an approximating distribution that reflects two crucial features of the finite-agent market: For one, the limiting distribution should allow for a nontrivial fraction of the population to remain unmatched. On the other hand, we need to strike the right balance between the magnitude of observed characteristics and idiosyncratic taste shocks so that in the limit the joint distribution of matched characteristics of men and women does not degenerate to a matching rule that is deterministic or independent of observed characteristics. We show below that imposing these two requirements simultaneously results in specific rates for payoff parameters along the limiting sequence. Specifically, the limiting distribution will have the first property only if the outside option is made more attractive at exactly the right rate as the size of the market grows. The second condition concerns the scale of the distribution of unobserved heterogeneity, where the rate of increase in the variance of idiosyncratic taste shocks depends on the tail behavior of its standardized distribution.

**Related Literature.** Most recent empirical work and identification analysis for matching markets, including Choo and Siow (2006), Fox (2010) and Galichon and Salanié (2012), has been focussing on the transferable utility (TU) framework. The NTU case analyzed in this paper is qualitatively different from models assuming TU in that stable matchings do
not necessarily maximize joint surplus across matched pairs. Specifically, in the TU case, pairwise stable matchings are typically unique, whereas in the NTU case the number of distinct stable matchings can be very large. Nonetheless, we find a similar representation of matching frequencies in terms of a surplus measure at the level of the matched pair. That new surplus measure does not have a straightforward economic interpretation the absence of transfers or a numéraire but is a sufficient parameter for predictions or policy counterfactuals. Our approach also differs in that we assume that individuals’ preference rankings are over individual potential spouses on the other side of the market. In contrast, previous work by Choo and Siow (2006), Galichon and Salanié (2012), and Hsieh (2012) assumes that unobserved taste shifters are group-specific, so that agents would be indifferent between potential spouses of the same observable type.

For the NTU case, Echenique, Lee, and Shum (2010) and Echenique, Lee, Shum, and Yenmez (2012) consider inference based on implications of matching stability assuming that agents’ types are discrete and fully observed by the econometrician. Uetake and Watanabe (2012) derive bounds on matching probabilities from the (necessary) pairwise stability conditions, and also extend the concept of pairwise stability to allow for externalities in the matching game. Their work is complementary to ours in that we rely on the number of agents in the matching market being large. We derive the limiting distribution for the model which can be used for (efficient) likelihood-based inference rather than bounds that need not be sharp. Logan, Hoff, and Newton (2008) estimate a NTU model for a matching market using a Bayesian approach, though they do not explicitly account for the possibility of multiple pairwise stable outcomes.

Pakes, Porter, Ho, and Ishii (2006), and Baccara, Imrohoroglu, Wilson, and Yariv (2012) estimate matching games via inequality restrictions on the conditional mean or median of payoff functions derived from necessary conditions for optimal choice. Instead, we model the full distribution of payoffs and match characteristics. In general, our approach requires some conditions on the distribution of unobserved heterogeneity (most importantly independence from observed characteristics and across agents), but on the other hand that knowledge permits to compute policy-relevant counterfactuals (e.g. conditional choice probabilities) based on estimated parameters and the asymptotic approximation to the distribution of matched characteristics.

A Matching market can also be viewed as a network formation game in which links are undirected, and each agent can form at most one link. For a more general case, Christakis, Fowler, Imbens, and Kalyanaraman (2010) use MCMC techniques to estimate a model of strategic network formation. Their analysis assumes a stochastic model in which decisions
on whether to establish or maintain a link are made sequentially by myopic agents instead of a static solution concept.\footnote{While for certain classes of games, adaptive processes of this kind are known to converge to stable outcomes according to some static solution concept e.g. for matching markets (see Roth and Sotomayor (1990) and Roth and Vande Vate (1990)) or games of strategic complementarities (see Milgrom and Roberts (1990)), there is not a perfect equivalence. Rather, randomization over starting points or meeting protocols would result in a mixture over static equilibrium outcomes rather than the full set of distributions that could be generated by the model under different assumptions on equilibrium selection.}

Our main approach towards identification and estimation is to work with a more tractable large-sample approximation to the distribution of observable characteristics at the level of pairs resulting from a stable matching. Our formal derivations for the limiting matching games rely on insights from Dagsvik (2000)’s analysis of aggregate matchings. Specifically, Dagsvik (2000) showed convergence of the distribution of matchings with no observable attributes under the assumption that individual taste shifters are extreme value type-I distributed and private knowledge, and characterized the limiting distribution for discrete attributes, assuming convergence. In contrast to our analysis, his model imposes a market clearing condition with respect to type-specific supply and demand probabilities as a solution concept, whereas we assume pairwise stability given the realized values of non-transferable utilities. We also extend his arguments to the empirically relevant case of continuous covariates, establishing uniqueness of the limiting distribution of observable characteristics in matched pairs. Previous characterizations of the distribution of match characteristics in both the TU and the NTU setting were limited to the case of finitely many observable types (this includes the results by Dagsvik (2000), Choo and Siow (2006), Galichon and Salanié (2012), Hsieh (2012), and Graham (2013)). To our knowledge the only exception is Dupuy and Galichon (2013)’s recent extension to Choo and Siow (2006)’s characterization of equilibrium in the TU matching model.

Agarwal (2012) and Azevedo and Leshno (2012) consider limiting distributions as the number of individuals on one side of the market grows. Interestingly, in both models, the stable matching is unique in the limit. Also, Decker, Lieb, McCann, and Stephens (2013) give a representation of equilibrium in the transferable utilities with finitely many types in terms of the share of unmarried agents of each type, and show uniqueness of the resulting matching equilibrium. There are some parallels to our result on asymptotic uniqueness for the NTU case with individual rather than type-specific heterogeneity - in particular, our fixed-point representation of the limiting distribution is in terms of inclusive value functions, which are linked to the type-specific shares of unmarried agents by a one-to-one transformation. In our analysis the number of agents on both sides of the market grows, and we show that the limiting distribution of matched characteristics is unique, though the number of distinct stable matchings supporting that distribution grows at a fast rate. Those stable matchings will in general differ from the agents’ perspective even in the limit - both in terms

of welfare and the identity of their realized match - and our uniqueness result only concerns observational equivalence from the econometrician’s perspective.

The implications of the independence of irrelevant alternatives (IIA) property for discrete choice models with a finite, or countably infinite number of alternatives have been analyzed by Luce (1959), McFadden (1974), Yellott (1977), Cosslett (1988), Resnick and Roy (1991), and Dagsvik (1994). We show that for the NTU model, the pairwise stability conditions translate into a discrete choice problem with unobserved and endogenous choice sets containing a large number of alternatives, where IIA arises as a limiting property of conditional choice probabilities, which greatly simplifies our analysis.

Overview and Notation. The next section develops a random utility model for matching preferences and states the main assumptions on the economic primitives for the matching market, where we assume pairwise stability as a solution concept. Section 3 contains our main formal results establishing convergence of the distribution of matching outcomes in the finite economy and characterizing its limit. Section 4 discusses the main practical implications of that limiting results, most importantly identification and estimation of payoff parameters from the limiting distribution. Section 5 proposes an extension that allows for exogenous search frictions, and discusses other qualitative and quantitative properties of the limiting model. Section 6 presents Monte Carlo evidence on the performance of the many-agent approximation for prediction and estimation.

We use standard “little-o” / “big-O” notation to denote orders of convergence for deterministic sequences, and convergence in probability for random sequences. We also write $a_n \asymp b_n$ if $\lim_n a_n/b_n = 1$. We also use the abbreviations a.s. and w.p.a.1 for the qualifiers “almost surely” and “with probability approaching one.”

2. Model Description

The researcher observes data on matching outcomes from one or several markets, where we denote the number of women and men in the market with $n_w$ and $n_m$, respectively. We assume that the data set contains variables $x_i$ and $z_j$ that have information on some of women $i$’s and man $j$’s characteristics, where both vectors may contain discrete and continuous variates. The respective marginal distributions of $x_i$ and $z_j$ in the population are given by the p.d.f.s $w(x)$ and $m(z)$, and we denote the supports of $x_i$ and $z_j$ with $\mathcal{X}$ and $\mathcal{Z}$, respectively. Furthermore, we observe whether man $j$ and woman $i$ are matched, and which individuals remain single. Specifically, we use $\mu_w(i)$ and $\mu_m(j)$ to denote woman $i$’s, and man $j$’s spouse, respectively, under the matching $\mu$. Each individual can marry a person of the opposite sex or choose to remain single.

2.1. Types and Preferences. We consider a matching model with non-transferable utilities (NTU), where preferences over spouses are given by the latent random utility functions
of the form

\[ U_{ij} = U(x_i, z_j) + \sigma \eta_{ij} \]
\[ V_{ji} = V(z_j, x_i) + \sigma \zeta_{ji} \]

(2.1)

for \( i = 1, \ldots, n_W \) and \( j = 1, \ldots, n_M \). The random utility for the outside option - i.e. of remaining single - is specified as

\[ U_{i0} = 0 + \sigma \max_{k=1,\ldots,J} \{ \eta_{i0,k} \} \]
\[ V_{j0} = 0 + \sigma \max_{k=1,\ldots,J} \{ \zeta_{j0,k} \} \]

(2.2)

where \( J \) increases at a rate to be specified below, and the scale parameter \( \sigma \) may also depend on market size.

In our setup, payoffs have “systematic” components \( U(x_i, z_j) \) and \( V(z_j, x_i) \) that are a function of individual \( i \) and \( j \)’s characteristics \( x_i, \) and \( z_j, \) respectively, and the “idiosyncratic” components \( \eta_{ij}, \zeta_{ji} \) and \( \eta_{i0,k}, \zeta_{j0,k} \) are i.i.d. draws from a known distribution that are independent of \( x_1, x_2, \ldots \) and \( z_1, z_2, \ldots \). For our derivation of the limiting distribution, we do not explicitly distinguish between components of \( x_i \) and \( z_j \) that are observed or unobserved. The central difference between the roles of the “systematic” and the “idiosyncratic” parts of the payoff functions in our model is that random taste shifters \( \eta_{ij} \) and \( \zeta_{ji} \) are assumed to be independent across \( i, j \) and therefore do not induce correlation in preferences across agents. The appropriate sequence for \( \sigma \) will generally depend on the shape of the tails of \( G(\cdot) \) and will be discussed below.

The model differs from the assumptions in previous work by Choo and Siow (2006) and Galichon and Salanié (2012) in that the idiosyncratic taste shocks for woman \( i \) and man \( j \), \( \eta_{ij} \) and \( \zeta_{ji} \), respectively, are individual-specific with respect to potential spouses \( j = 1, \ldots, n_m \) and \( i = 1, \ldots, n_w \) rather than only allowing for heterogeneity in tastes over a finite number of observable characteristics. In that aspect, our random utility model is similar to that in Dagsvik (2000), though in addition our setup allows for the systematic part of the random utility functions to depend on continuous characteristics, and we do not assume a particular distribution for the idiosyncratic taste shifters.

The rationale for modeling the outside option as the maximum of \( J \) independent draws for the idiosyncratic taste shifters is that as the market grows, the typical agent can choose from an increasing number of potential spouses. Since in our setup the shocks \( \eta_{ij} \) and \( \zeta_{ji} \) generally have unbounded support, any alternative with a fixed utility level will eventually be dominated by one of the largest draws for the increasing set of potential matching partners. Hence, by allowing the agent to sample an increasing number of independent draws for the outside option, it can be kept sufficiently attractive to ensure that the share of unmatched agents remains stable along the sequence. Alternatively, one could model the outside option as \( \tilde{U}_{i0} = \log J + \sigma \eta_{i0} \), where \( \eta_{i0} \) is a single draw from the distribution \( G(\eta) \), as e.g. in Dagsvik (2000). Preliminary calculations suggest that both approaches lead to equivalent results, but
the formulation with “multiple outside options” is more convenient for our derivations than the “location shift” version.

The assumption of non-transferable utility makes our results applicable to markets in which transfers between matching partners are restricted or ruled out altogether. Institutional restrictions on transfers or side payments are often motivated by ethical or distributional concerns, and are common e.g. for assigning students to schools or colleges or residents to hospitals in the medical match. Note that our assumptions allow for transfers that are deterministic functions of characteristics $x_i, z_j$, in which case we can interpret $U(x_i, z_j)$ and $V(z_j, x_i)$ as the systematic parts of payoffs net of transfers. For example, employment contracts between workers and firms may be subject to collective bargaining agreements, which stipulate a fixed wage given the job description and worker’s education, experience, or tenure at the firm.

Throughout the paper, we will maintain that the deterministic parts of random payoffs satisfy certain uniform bounds and smoothness restrictions:

**Assumption 2.1. (Systematic Part of Payoffs)** The functions $|U(x, z)| \leq \bar{U} < \infty$ and $|V(z, x)| \leq \bar{V} < \infty$ are uniformly bounded in absolute value and continuous in $X \times Z$. Furthermore, at all $(x', z')' \in X \times Z$ the functions $U(x, z)$, and $V(z, x)$ are $p \geq 1$ times differentiable with uniformly bounded partial derivatives.

For notational simplicity, we do not distinguish between components of $x, z$ that are continuous and those which only take discrete values. The differentiability requirements could be weakened to apply only to the continuous arguments of $U(\cdot)$ and $V(\cdot)$, holding the discrete components fixed. Also notice that the uniform bound on the absolute value of systematic parts would require that individual preference rankings may not be lexicographic in certain observables.\(^2\)

We next state our assumptions on the distribution of unobserved taste shifters. Most importantly, we impose sufficient conditions for the distribution of $\max_j \eta_{ij}$ to belong to the domain of attraction of the extreme-value type I (Gumbel) distribution. Following Resnick (1987), we say that the upper tail of the distribution $G(\eta)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the c.d.f. satisfies

$$
\lim_{s \to \infty} \frac{1 - G(s + a(s)v)}{1 - G(s)} = e^{-v}
$$

for all $v \in \mathbb{R}$. We are furthermore going to restrict our attention to distributions for which the auxiliary function can be chosen as $a(s) := \frac{1 - G(s)}{g(s)}$. We can now state our main assumption on the distribution of the idiosyncratic part of payoffs:

\(^2\)Lexicographic priority rankings may arise e.g. in assigning students to schools if schools first grant admission to all applicants that meet certain requirements (e.g. siblings attending the same school or living within walking distance), and randomize priorities only for the remaining slots and applicants.
Assumption 2.2. (Idiosyncratic Part of Payoffs) \( \eta_{ij} \) and \( \zeta_{ji} \) are i.i.d. draws from the distributions \( G(s) \), and are independent of \( x_i, z_j \), where (i) the c.d.f. \( G(s) \) is absolutely continuous with density \( g(s) \), and (ii) the upper tail of the distribution \( G(s) \) is of type I with auxiliary function \( a(s) := \frac{1-G(s)}{g(s)} \).

The assumption of a continuous distribution for \( \eta \) and \( \zeta \) in part (i) is fairly standard, and as discussed above, part (ii) of the assumption ensures that the distribution of unobserved taste shocks belongs to the domain of attraction of the extreme-value type-I (Gumbel) distribution. While we do not give more primitive conditions, it is known from extreme-value theory that the tail condition in part (ii) is satisfied for most parametric specifications for the distribution of \( \eta_{ij} \) that are commonly used in discrete choice models. For example, if \( G \) is the Extreme-Value Type-I distribution or the Gamma distribution, then this condition holds with auxiliary function \( a(s) = 1 \). For standard normal taste shifters, part (ii) holds with auxiliary function \( a(s) = \frac{1}{s} \).

As we already pointed out before, the independence assumptions for the idiosyncratic taste shifters across alternatives is a strong, but nevertheless very common restriction in the discrete choice literature. Conditional independence across individuals and alternatives implies that any correlations in preferences for different alternatives must be due to the systematic part of random utility functions.

It should also be noted that the i.i.d. assumption is the only substantive distinction between our statistical model and the standard transferable utility (TU) framework, as well as many-to-one matchings. While we can easily extend our framework to allow for transfers that are deterministic functions of \( x_i, z_j \) as argued before, the assumption of conditional independence of net payoffs \( U_{ij} \) and \( V_{ji} \) cannot be reconciled with the standard transferable utility model.\(^4\) In addition, independence of \( \eta_{ij} \) across \( i \) and \( j \) also rules out most common specifications for “many to one” matchings in which some agents may be matched with several individuals of the opposite side of the market unless taste shifters are drawn independently for each possible link that agent may form.\(^5\)

2.2. Pairwise Stability. One standard solution concept for matching markets is pairwise stability: A matching \( \mu^* \) is pairwise stable given the preferences \( U_{ij}, V_{ji} \), where \( i = 1, \ldots, n_w, j = 1, \ldots, n_m \), if every individual prefers her/his spouse under \( \mu^* \) to any other

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\(^3\)See Resnick (1987), pages 42-43.

\(^4\)Generally, transfers resulting from splitting the match surplus mechanically depend on the idiosyncratic taste shifters and therefore generally result in net payoffs \( U_{ij} \) and \( V_{ji} \) that are correlated conditional on \( x_i, z_j \).

\(^5\)For example, in a typical implementation of a mechanism assigning students to schools a given student’s priority conditional on other systematic factors (e.g. whether a student lives within walking distance or has siblings that are already enrolled in the school) is randomized only once across all schools and/or seats in any given school. Our results would apply to a problem of this type if instead priorities were randomized independently across schools and slots.
achievable partner. I.e. $\mu^*$ must satisfy the conditions (i) if $U_{ij} > U_{i\mu^*_n(i)}$, then $V_{j\mu^*_m(j)} \geq V_{ji}$, and (ii) if $V_{ji} > V_{j\mu^*_m(j)}$, then $U_{i\mu^*_n(i)} \geq U_{ij}$.

The problem of pairwise stable matchings with non-transferable utility has been studied extensively and is well-understood from a theoretical perspective. If preferences are strict, a stable matching always exists, and in general the set of stable matchings has a maximal element $\mu^W$ and a minimal element $\mu^M$ with respect to the preferences of the female side which we refer to as the W-preferred and M-preferred stable matching, respectively, whereas the preferences over matchings on the male side are exactly opposed. Specifically, for any stable matching $\mu^*$, the W-preferred stable matching satisfies $U_{i\mu^*_n(i)} \geq U_{i\mu^*_m(i)}$ and $V_{j\mu^*_m(j)} \leq V_{j\mu^*_n(j)}$, and for the M-preferred stable matching we always have $U_{i\mu^*_n(i)} \leq U_{i\mu^*_m(i)}$ and $V_{j\mu^*_m(j)} \geq V_{j\mu^*_n(j)}$. The M-preferred and W-preferred stable matching can be found in polynomial computing time using the well-known Gale-Shapley algorithm. The number of distinct stable matchings for a typical realization of preferences increases exponentially in the number of individuals on each side of the market (see Theorem 3.19 in Roth and Sotomayor (1990)).

Pairwise stable matchings may arise in centralized matching markets that employ a variant of the Gale-Shapley algorithm to assign matching partners. Roth and Vande Vate (1990) show that randomized myopic tâtonnement processes converge to pairwise stable matchings with probability one if allowed to continue indefinitely, which may also justify matching stability as a suitable empirical model for decentralized markets. It is also important to notice that pairwise stability as a solution concept does not require that agents have perfect knowledge of all participants’ preferences, but each agent only needs to know which matching partners are available to him or her. In section 6, we also give a straightforward extension to the main model which allows for agents only to be aware of a random subset of potential matching partners.

For our analysis it is useful to translate the pairwise stability conditions into a discrete-choice problem at the individual level: Given a matching $\mu$, we let the set $M_i \equiv M_i[\mu] \subset \{1, \ldots, n_M\}$ denote the set of men $j$ preferring woman $i$ over their current match, $\mu_m(j)$, i.e.

$$j \in M_i[\mu] \text{ if and only if } V_{ji} \geq V_{j\mu_m(j)}$$

We call $M_i[\mu]$ the set of men available to woman $i$ under the matching $\mu$, or woman $i$’s opportunity set. As a notational convention, we do not treat the outside option as an element of $M_i$, i.e. $0 \notin M_i$, notwithstanding the fact that the outside option is always available to agent $i$ in our setup. Similarly, we define man $j$’s opportunity set $W_j \equiv W_j[\mu]$ as

$$i \in W_j[\mu] \text{ if and only if } U_{ij} \geq U_{i\mu_n(i)}$$

\textsuperscript{6}See Roth and Sotomayor (1990) for a synthesis of the classical results on the problem.
For any opportunity sets $M$ and $W$, we define indirect random utility

$$U_i^*(M) := \max_{j \in M \cup \{0\}} U_{ij} \quad \text{and} \quad V_j^*(W) := \max_{i \in W \cup \{0\}} V_{ji}$$

Using this notation, we can rewrite the pairwise stability condition in terms of individually optimal choices from the opportunity sets arising from a matching $\mu$.

**Lemma 2.1.** Assuming that all preferences are strict, a matching $\mu$ is pairwise stable if and only if

$$U_{i\mu_i(w)}(i) \geq U_i^*(M_i[\mu]) \quad \text{and} \quad V_{j\mu_j(m)}(j) \geq V_j^*(W_j[\mu])$$

for all $i = 1, \ldots, n_w$ and $j = 1, \ldots, n_m$.

See the appendix for a proof of this lemma.

### 2.3. Asymptotic Sequence.

Before developing the asymptotic argument, we also need to specify the approximating sequence of markets. Specifically, we want the approximation to retain several qualitative features of the finite-agent market: for one, the share of single individuals should not degenerate to one or zero. Furthermore, we want the systematic parts of payoffs to remain predictive for match probabilities in the limit, although the joint distribution of male/female match characteristics should also not be degenerate in the limit. For the first requirement, it is necessary to increase the payoff from outside option as the number of available alternatives grows, whereas to balance the relative scales of the systematic and idiosyncratic parts we have to choose the scale parameter $\sigma \equiv \sigma_n$ at an appropriate rate.

**Assumption 2.3. (Market Size)** (i) The size of a given market is governed by $n = 1, 2, \ldots$ with the number of men and women $n_m = \lceil n \exp\{\gamma_m\} \rceil$ and $n_w = \lceil n \exp\{\gamma_w\} \rceil$, where $\gamma_w$ and $\gamma_m$ are bounded in absolute value across markets and $\lceil x \rceil$ denotes the value of $x$ rounded to the closest integer. (ii) The size of the outside option is $J = \lceil n^{1/2} \rceil$, and (iii) the scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1}\left(1 - \frac{1}{\sqrt{n}}\right)$, and $a(s)$ is the auxiliary function specified in Assumption 2.2 (ii).

Part (i) requires that the number of men and women in each market is of comparable magnitude. The rate for $J$ in part (ii) is chosen to ensure that the share of unmatched agents is bounded away from zero and one along the sequence.\(^7\)

The construction of the sequence $\sigma_n$ in part (iii) implies a scale normalization for the systematic parts $\tilde{U}_{ij} := U(x_i, z_j)$, and is chosen as to balance the relative magnitude for the respective effects of observed and unobserved taste shifters on choices as $n$ grows large. Specifically, for an alternative rate $\tilde{\sigma}_n$ such that $\tilde{\sigma}_n a(b_n) \to 0$, the systematic parts of payoffs

\(^7\)Note also that choosing $\tilde{J}_n = \lceil \alpha n^{1/2} \rceil$ for a given choice of $\alpha > 0$ would be equivalent to the original rate $J_n = \lceil \sqrt{n} \rceil$ with a different value for the intercept of the random utility from the outside option, so that our implicit choice $\alpha = 1$ is only a location normalization.
\( \tilde{U}_1, \ldots, \tilde{U}_{in_m} \) become perfect predictors for choices as \( n \) grows large, whereas if \( \tilde{\sigma}_n a(b_n) \rightarrow \infty \), the systematic parts become uninformative in the limit. For example, if \( G(\eta) \) has very thin tails, the distribution of the maximum of \( J \) i.i.d. draws from \( G(\cdot) \) becomes degenerate at a deterministic drifting sequence as \( J \) grows, and it is therefore necessary to increase the scale parameter \( \sigma \) in order for the scale of the maximum of idiosyncratic taste shifters to remain of the same order as differences in the systematic part. Specifically, if \( \eta_{ij} \sim \Lambda(\eta) \), the extreme-value type I (or Gumbel) distribution, then we choose \( b_n \approx \frac{1}{2} \log n \) and \( \sigma_n = 1 \). For \( \eta_{ij} \sim N(0, \sigma^2) \), it follows from known results from extreme value theory that the constants can be chosen as \( b_n \approx \sigma \sqrt{W(n/2\pi)} \approx \sigma \sqrt{\log n} \) and \( \sigma_n \approx \frac{b_n}{\sigma^2} \), where \( W(x) \) is the Lambert-W (product log) function, and for Gamma-distributed \( \eta_{ij} \), \( b_n \approx \log n \) and \( \sigma_n = 1 \).

2.4. Inclusive Values. The main difficulty in solving for the distribution of stable matchings consists in the fact that the set of available spouses \( M_i \) and \( W_j \) for each individual is not observed by the researcher, and determined endogenously in the market. We show that for a broad class of commonly used specifications for random utility models (see Assumption 2.2 below), the conditional choice probabilities converge to those implied by the conditional logit (extreme-value type I) model in the limit. In that case, the composition and size of the set of available spouses affects the conditional choice probabilities only through the inclusive value, which is a scalar parameter summarizing the systematic components of payoffs for the available options, see Luce (1959), McFadden (1974), and Dagsvik (1994).

Specifically, we define the inclusive values for woman \( i \)'s opportunity set \( M \), \( I_{wi}[M] \), and for man \( j \)'s opportunity set \( W \), \( I_{mj}[W] \) respectively, as

\[
I_{wi}[M] := \frac{1}{n^{1/2}} \sum_{j \in M} \exp \{ U(x_i, z_j) \} \\
I_{mj}[W] := \frac{1}{n^{1/2}} \sum_{i \in W} \exp \{ V(z_j, x_i) \}
\]

where \( n \) was specified in Assumption 2.3. The normalization by \( n^{-1/2} \) is arbitrary at this point, but will be convenient for the asymptotic analysis. Specifically, we show below that the size of a typical participant’s opportunity set grows at a rate proportional to \( n^{1/2} \). Furthermore, the inclusive value \( I_{mj}[W] \) has the form of a sample average over potential spouses, and we show below that it converges to its conditional expectation given \( x_i \). In particular, in terms of the implied inclusive values (and resulting conditional choice probabilities) two women with the same observable characteristics \( x \) will face very similar matching opportunities as the market grows thick.

\[8\] See Resnick (1987), pages 71-72.
LARGE MATCHING MARKETS

Note also that we depart from the usual definition of the inclusive value as the conditional expectation of woman $i$’s indirect utility from a choice set $M$,

$$
\mathbb{E} \left[ \max_{j \in M \cup \{0\}} U_{ij} \bigg| x_i, (z_j)_{j \in M} \right] = \log \left( n^{1/2} + \sum_{j \in M} \exp \{ U(x_i, z_j) \} \right) + \kappa
$$

$$
= \log(1 + I_{wi}[M]) + \frac{1}{2} \log n + \kappa
$$

where $\kappa$ is Euler’s constant, see e.g. McFadden (1974). While there is a one-to-one transformation between the usual definition of the inclusive value and $I_{wi}[M]$, we find it more convenient to describe our results in terms of $I_{wi}[M]$ and $I_{mj}[w]$ and will apply the term “inclusive value” directly to those variables. Note also that in the context of the conditional logit model, the relationship between $I_{wi}[M]$ and expected indirect utility gives inclusive values a straightforward interpretation as a surplus measure that can be used for welfare analyses.

As a preview of our formal results, we find that in that limit, the respective inclusive values of the (endogenous) opportunity sets $M_i$ and $W_j$ are sufficient statistics with respect to the conditional choice probabilities, and pairwise stability is shown to be asymptotically equivalent to a fixed-point condition on the inclusive values. Therefore, the extreme-value limit of the conditional choice probabilities implies a substantive simplification of the latent state space of the matching model.

2.5. Matching Frequency and Sampling Distribution. The main object of interest in this paper is the distribution of matched characteristics from a stable matching $\mu^*$. Conditional on the marginal distributions of characteristics $w(x)$ and $m(z)$, the realized stable matching is generally random - for one, individual preferences also depend on the unobserved payoff shifters $\eta_{ij}, \zeta_{ji}$, and furthermore the observed matching may be selected at random from the set of pairwise stable matchings.

Given a random matching $\mu$, we define the matching frequency distribution as the expected number of matched pairs of men and women of observable types $z$ and $x$, respectively. Specifically, let

$$
F_n(x, z|\mu) := \frac{1}{n} \sum_{i=0}^{n_w} \sum_{j=0}^{n_m} P(x_i \leq x, z_j \leq z, j \in \mu_w(i) \text{ or } i \in \mu_m(j))
$$

where, as a convention, $\mu_w(0)$ denotes the set of men that are single under the matching $\mu$, and missing values of $x$ and $z$ for unmatched agents are coded as $+\infty$. The normalization by $\frac{1}{n}$ is arbitrary at this point but ensures that $F_n(x, z|\mu^*_n)$ converges to a non-degenerate limit along a sequence of arbitrarily selected stable matchings $\mu^*_n$, a measure which we denote with $F(x, z)$. We also denote the corresponding empirical matching frequencies given the realized
matching with

$$\hat{F}_n(x, z) := \frac{1}{n} \sum_{i=0}^{n_w} \sum_{j \in \mu_w(i)} \mathbb{1} \{x_i \leq x, z_j \leq z\}$$

Note also that the empirical matching frequencies could equivalently be defined as

$$\hat{F}_n(x, z | \mu) := \frac{1}{n} \sum_{j=0}^{n_m} \sum_{i \in \mu_m(j)} \mathbb{1} \{x_i \leq x, z_j \leq z\}$$

where by the definition of a matching, $\mu_m(j)$ and $\mu_w(i)$ are singleton for all values of $i, j$ except zero.

The limiting measure $F(\cdot)$ is in general not a proper probability distribution but integrates to the mass of couples that form in equilibrium. In general this will be a value between $\exp\{\max\{\gamma_w, \gamma_m\}\}$ and $\exp\{\gamma_w\} + \exp\{\gamma_m\}$ that may be different from one. However, we can use this measure to derive the sampling distributions of matched characteristics under any given sampling protocol, e.g. depending on whether the researcher samples individuals or couples from the population. Implications for identification and estimation of preference parameters will be discussed in more detail in section 5.

Let $f(x, z)$ denote the joint density of observable characteristics of a matched pair, defined as the Radon Nikodym derivative of the measure $F$. As a convention we let $f(x, \ast)$ and $f(\ast, z)$ denote the density of characteristics among unmatched women and men, respectively. In particular, the following relations hold:

$$\int_Z f(x, z)dz + f(x, \ast) = w(x) \exp\{\gamma_w\} \quad \text{and} \quad \int_X f(x, z)dx + f(\ast, z) = m(z) \exp\{\gamma_m\}$$

For identification and estimation, we assume that we observe a sample of $K$ couples ("households") $k = 1, \ldots, K$, where $w(k)$ and $m(k)$ give the respective indices of the wife and the husband, where $m(k) = 0$ if the $k$th unit represents of a single woman and $w(k) = 0$ if we observe a single man. Note that any sampling strategy for selecting households corresponds to a different protocol for sampling from the matching frequency distribution, and we can obtain the density $h(x, z)$ of the resulting sampling distribution of $(x_{w(k)}, z_{m(k)})$ from the density $f(x, z)$.

First, consider the case of a random sample of individuals, where each man and woman may be selected for a survey with the same probability. The resulting sample reports the spouses’ characteristics $(x_{w(k)}, z_{m(k)})$ for the $k$th unit. If the selected individual is single, we only observe his own characteristics, and the spousal characteristics are coded as missing. However, assuming sampling with replacement, the probability that a married man is included in the survey equals the probability that he is selected as a primary respondent plus the probability that his spouse is. In contrast, a single man is only contacted if he is selected
as a primary respondent. Hence, the sampling distribution is given by the p.d.f.

\[ h_1(x, z) = \frac{2f(x, z)}{\exp{\gamma_w} + \exp{\gamma_m}} \]  
\[ h_1(x, *) = \frac{f(x, *)}{\exp{\gamma_w} + \exp{\gamma_m}}, \quad h_1(*, z) = \frac{f(*, z)}{\exp{\gamma_w} + \exp{\gamma_m}} \]  

Alternatively, if the survey design draws at random from the population couples ("households"), including singles, then the sampling distribution is characterized by the p.d.f.

\[ h_2(x, z) = \frac{f(x, z)}{\exp{\gamma_w} + \exp{\gamma_m} - \int_{X \times Z} f(s, t) dt ds} \]

for all \( x \in X \cup \{\ast\} \) and \( z \in Z \cup \{\ast\} \). This discussion could easily be extended to cases in which a survey contains sampling weights that allow us to reconstruct an weighted sample with comparable properties. However, it is important that for our analysis the matched pair is not the unit of observation, but endogenous to the model. In the remainder of the paper, we describe matching outcomes primarily in terms of the implied matching frequency distribution rather than a sampling distribution for notational convenience.

3. Convergence of the Finite Economy

This section establishes convergence of the distributions generated by stable matchings in the finite economy and derives their common limit. Specifically, Theorem 3.1 establishes that the limiting distribution is uniquely defined and does not depend on how a pairwise stable matching is selected in the finite economy. We give the main convergence results in Theorem 3.2 and Corollary 3.1, and a qualitative outline of the main technical arguments.

The asymptotic argument will proceed in four main steps: First, we show convergence of conditional choice probabilities (CCP) to CCPs generated by the Extreme-Value Type-I taste shifters, under the assumption that taste shifters \( \eta_{ij} \) are independent from the equilibrium opportunity sets \( W_i \) and \( M_j \). Secondly, we demonstrate that dependence of taste shifters and opportunity sets is negligible for CCPs when \( n \) is large. Hence we can approximate choice probabilities using the inclusive values. In a third step, we establish that the inclusive values \( I_w[M_i] \) and \( I_m[W_j] \) are approximated by the inclusive value functions \( \hat{\Gamma}_w(x_i) \) and \( \hat{\Gamma}_m(z_j) \) to be defined below. Finally we show that the inclusive value functions \( \hat{\Gamma}_w \) and \( \hat{\Gamma}_m \) corresponding to stable matchings are approximate solutions to a sample analog of the fixed-point problem (3.5). Uniform convergence of the sample fixed point mapping to its population version then implies that the inclusive values converge in probability to the limits \( \Gamma_w^* \) and \( \Gamma_m^* \). In order to limit the amount of additional notation in the main text, we only state partial or simplified versions of auxiliary results in this section, and relegate the more general statements and proofs to the appendix.
3.1. **Convergence to Logit CCPs.** For the first step, Lemma B.1 in the appendix establishes that if the distribution of $\eta$ is in the domain of attraction of the extreme-value type-I distribution, then for large sets of alternatives, the implied conditional choice probabilities can be approximated by those implied by the Logit model. For expositional clarity, the main text only states the main approximation result in terms of the “unilateral” decision problem of a single agent facing a choice over an increasing set of alternatives.

**Lemma 3.1.** Suppose that Assumptions 2.1, 2.2, and 2.3 hold, and that $z_1, \ldots, z_J$ are $i.i.d$ draws from a distribution $M(z)$ with p.d.f. $m(z)$. Then as $J \to \infty$,

$$P(U_{i0} \geq U_{ik}, k = 0, \ldots, J) \to \frac{1}{1 + \int \exp\{U(x_i, s)\} m(s) ds}$$

$$JP(U_{ij} \geq U_{ik}, k = 0, \ldots, J | z_j = z) \to \frac{\exp\{U(x_i, z)\}}{1 + \int \exp\{U(x_i, s)\} m(s) ds}$$

almost surely for any fixed $j = 1, 2, \ldots, J$.

See the appendix for a proof. Note that the limits on the right-hand side correspond to the choice probabilities for random sets of alternatives under the IIA assumption which were derived by Dagsvik (1994). It is important to note that the rate of convergence to the limiting choice probabilities depends crucially on the shape of the tails of $G(s)$. While for some choices for $G(s)$, convergence may be very fast, in the case of the standard normal distribution the rate of approximation for the c.d.f. of the maximum is as slow at $1/\log n$ (see e.g. Hall (1979)). Hence, Lemma 3.1 suggests that CCPs resulting from extreme-value type-I taste shifters can be viewed as a reference case for modeling choice among large sets of alternatives even if convergence to that limit may be very slow for alternative specifications. The extreme value approximation to conditional choice probabilities also requires that individuals’ opportunity sets grow in size as the market gets large. We show in Lemma B.2 in the appendix that the opportunity sets $W_i$ and $M_j$ do indeed grow at the rate $n^{1/2}$ with common bounds that hold for all individuals simultaneously with probability approaching one.

3.2. **Endogeneity of Opportunity Sets.** For the second step, we need to address that woman $i$’s opportunity set is in general endogenous with respect to her own taste shifters. Define the indicator variable $D^*_{ij} := \mathbb{1}\{j \in M^*_i\}$ which is equal to 1 if man $j$ is available to woman $i$ under the stable matching $\mu^*$, and zero otherwise. Similarly, we let $D^W_{ij}$ and $D^M_{ij}$ denote the corresponding availability dummies under the W-stable and the M-stable matching, respectively. We can then define the conditional p.d.f. of $\eta_i := (\eta_{i0}, \eta_{i1}, \ldots, \eta_{in_m})'$ given $D^W_i := (D^W_{i1}, \ldots, D^W_{in_m})'$ and $D^M_i := (D^M_{i1}, \ldots, D^M_{in_m})'$ by $g^W_{\eta|D^W}(\eta | D^W_i)$ and $g^M_{\eta|D^M}(\eta | D^M_i)$, respectively. Note that for a given realization of taste shifters, the two extremal matchings
are uniquely defined with probability one, so that the joint distribution of $\eta_i$ with $D_{ij}^W$ and $D_{ij}^M$ is indeed well-defined.

This second formal step establishes that endogeneity of woman $i$’s opportunity set with respect to her taste shifters $\eta_i$ vanishes in the sense that the conditional p.d.f. of $\eta_i$ given the availability indicators converges to the marginal distribution of $\eta_i$ with p.d.f. $g(\eta)$ as specified in Assumption 2.2.

**Lemma 3.2.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then (a) the conditional distributions for $\eta$ given $D_i^W$ and $D_i^M$, respectively, satisfy

$$\lim \limits_n \left| \frac{g_{\eta|D}(\eta|D_i^W)}{g_\eta(\eta)} - 1 \right| = \lim \limits_n \left| \frac{g_{\eta|D}(\eta|D_i^M)}{g_\eta(\eta)} - 1 \right| = 0$$

with probability approaching one as $n \to \infty$. The analogous results hold for the male side of the market.

This Lemma corresponds to part (a) of Lemma B.4, which is stated and proven in the appendix. The main argument of the proof considers an arbitrary change of $i$’s taste shifters $\eta_i$, and bounds the probability that this change results in a different value for any of the availability indicators $D_i^M$ or $D_i^W$ by a decreasing function in $n$. Such a shift in woman $i$’s preferences generally changes her availability to men in the market, but such a change is inconsequential if a given man hadn’t been available to $i$ to begin with. We show that this limits the degree through which changes in $i$’s preferences may percolate through the market to a sufficient degree that an indirect feedback on $i$’s opportunity set becomes increasingly unlikely as the size of the market grows.

It is important to point out that this convergence result does not rely on any assumptions regarding how the data generating process selects among the multiple stable matchings. Instead, the remainder of our argument is based on bounds on conditional choice probabilities and inclusive values from the M- and W-preferred stable matchings, respectively.

### 3.3. Law of Large Numbers for Inclusive Values.

For the third step, we show that woman $i$’s conditional choice probabilities given her opportunity set can be approximated using a state variable $\hat{\Gamma}_w(x_i)$ that depends only on her observable characteristics $x_i$. Specifically, we consider the inclusive values associated with the extremal matchings, where for the M-preferred matching, we denote

$$I_{wi}^M := I_{wi}[M_i^M] = n^{-1/2} \sum \exp \{ U(x_i, z_j) \}$$

$$I_{mj}^M := I_{mj}[W_j^M] = n^{-1/2} \sum \exp \{ V(z_j, x_i) \}$$
and we also write $I_{wi}^W$ and $I_{mj}^W$ for the inclusive values resulting from the W-preferred matching, and $I_{wi}^*$ and $I_{mj}^*$ for any other stable matching. In analogy to our definitions given exogenously fixed choice sets, we define the average inclusive value function under the M-preferred matching as

$$\hat{\Gamma}_w^M(x) := \frac{1}{n} \sum_{j=1}^{n_m} \exp\{U(x, z_j) + V(z_j, x)\} \frac{1}{1 + I_{mj}^M}$$

$$\hat{\Gamma}_m^M(z) := \frac{1}{n} \sum_{i=1}^{n_w} \exp\{U(x_i, z) + V(z, x_i)\} \frac{1}{1 + I_{wi}^M}$$

and, we define $\hat{\Gamma}_w^W(x)$, $\hat{\Gamma}_m^W(z)$, $\hat{\Gamma}_w^*(x)$, and $\hat{\Gamma}_m^*(z)$ in a similar manner for the W-preferred, or some generic matching, respectively. Since the opportunity sets $M_i^*$ and $W_j^*$ arising from any stable matching satisfy $M_i^M \subset M_i^* \subset M_i^W$ and $W_j^W \subset W_j^* \subset W_j^M$, we immediately obtain the relations

$$I_{wi}^M \leq I_{wi}^* \leq I_{wi}^W, \text{ and } I_{mj}^M \geq I_{mj}^* \geq I_{mj}^W$$

implying that

$$\hat{\Gamma}_w^M(x) \leq \hat{\Gamma}_w^*(x) \leq \hat{\Gamma}_w^W(x), \text{ and } \hat{\Gamma}_m^M(z) \geq \hat{\Gamma}_m^*(z) \geq \hat{\Gamma}_m^W(z)$$

for all values of $x$ and $z$, respectively. Hence, we can use the average inclusive value functions corresponding to the extremal matchings to bound those associated with any other stable matching.

Since $U(x, z)$ is bounded, we obtain

$$I_{wi}^M - \hat{\Gamma}_w^M(x_i) = \frac{1}{n^{1/2}} \sum_{j=1}^{n_m} \exp\{U(x_i, z_j)\} \left[ \mathbb{I}\{V_{ji} \geq V_{j}^M[W_j^M]\} - \Lambda_m(z_j, x_i; W_j^M) \right] + o_p(1)$$

where Lemma B.4 also implies that the random variables

$$v_{ji}(I_{mj}^M) := \mathbb{I}\{V_{ji} \geq V_{j}^M[W_j^M]\} - \Lambda_m(z_j, x_i; I_{mj}^M)$$

are approximately independent across $j = 1, \ldots, n_m$ conditional on $I_{mj}^M$. Hence, the difference $I_{wi}^M - \hat{\Gamma}_w^M(x_i)$ can be approximated as a weighted average of mean-zero random variables, where the pairwise correlations vanish sufficiently fast as $n$ grows. This argument is made precise in the following lemma:

**Lemma 3.3.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then,

$$I_{wi}^M \geq \hat{\Gamma}_{wi}^M(x_i) + o_p(1) \text{ and } I_{mj}^M \leq \hat{\Gamma}_{mn}^M(z_j) + o_p(1)$$

for all $i = 1, \ldots, n_w$ and $j = 1, \ldots, n_m$. The analogous result holds for the W-preferred matching.
This Lemma corresponds to part (a) of Lemma B.5, which is stated and proven in the appendix. Most importantly, this result allows us to approximate inclusive values as a function of observable characteristics alone.

3.4. Fixed-Point Representation. We then proceed to the fourth step of the main argument and derive an (approximate) fixed point representation for the inclusive values. Note that by convergence of $I^*_{mj}$ to $\Gamma^*_m(z_j)$ and the continuous mapping theorem we can write

$$\hat{\Gamma}_w(x_i) = \frac{1}{n} \sum_{j=1}^{n_m} \exp \left\{ \frac{U(x_i, z_j) + V(z_j, x_i)}{1 + \Gamma^*_m(z_j)} \right\} + o_p(1)$$

We next define the fixed point mapping

$$\hat{\Psi}_w[\Gamma_m](x) = \frac{1}{n} \sum_{j=1}^{n_m} \exp \left\{ \frac{U(x, z_j) + V(z_j, x)}{1 + \Gamma_m(z_j)} \right\}$$

$$\hat{\Psi}_m[\Gamma_w](z) = \frac{1}{n} \sum_{i=1}^{n_w} \exp \left\{ \frac{U(x_i, z) + V(z, x_i)}{1 + \Gamma_w(x_i)} \right\}$$ (3.1)

In that notation, we can characterize the pairwise stability conditions as the fixed-point problem

$$\hat{\Gamma}_m^* = \hat{\Psi}_m[\hat{\Gamma}_w^*] + o_p(1) \quad \text{and} \quad \hat{\Gamma}_w^* = \hat{\Psi}_w[\hat{\Gamma}_m^*] + o_p(1)$$ (3.2)

where, noting that $\hat{\Gamma}_m^*, \hat{\Gamma}_w^* \geq 0$ a.s., the remainder converges in probability to zero uniformly in $\Gamma_w, \Gamma_m$ by Lemma B.5 in the appendix and the continuous mapping theorem. In particular, the inclusive value functions for the two extremal matchings, $\hat{\Gamma}^M$ and $\hat{\Gamma}^W$ are solutions to the same fixed point problem.

3.5. Existence and Uniqueness of Fixed Point. In order to characterize the limits for $\hat{\Gamma}$, we define the population analogs of the equilibrium conditions in terms of the inclusive value functions $\Gamma_w(x)$ and $\Gamma_m(z)$,

$$\Gamma_w(x) = \int \frac{\exp\{U(x, s) + V(s, x) + \gamma_m\} m(s)}{1 + \Gamma_m(s)} ds$$

$$\Gamma_m(z) = \int \frac{\exp\{U(s, z) + V(z, s) + \gamma_w\} w(s)}{1 + \Gamma_w(s)} ds$$ (3.3)

We also define the operators $\Psi_w : \Gamma_w \mapsto \Gamma_m$ and $\Psi_m : \Gamma_m \mapsto \Gamma_w$ by

$$\Psi_w[\Gamma_m](x) := \int \frac{\exp\{U(x, s) + V(s, x) + \gamma_m\} m(s)}{1 + \Gamma_m(s)} ds$$

$$\Psi_m[\Gamma_w](z) := \int \frac{\exp\{U(s, z) + V(z, s) + \gamma_w\} w(s)}{1 + \Gamma_w(s)} ds$$ (3.4)

In that notation, we can rewrite the equilibrium conditions in (3.3) as the fixed-point problem

$$\Gamma_w^* = \Psi_w[\Gamma_m^*] \quad \text{and} \quad \Gamma_m^* = \Psi_m[\Gamma_w^*]$$ (3.5)
To simplify notation, in the following we will write \( \Gamma := (\Gamma_w, \Gamma_m) \) and \( \Psi[\Gamma] := (\Psi_w[\Gamma_m], \Psi_m[\Gamma_w]) \) with analogous expressions for the sample quantities \( \hat{\Gamma}_w, \hat{\Gamma}_m, \hat{\Psi}_w, \hat{\Psi}_m \). We then consider the equivalent problem

\[
\log \Gamma = \log \Psi[\Gamma]
\]

and show that under Assumption 2.1, the mapping \( \log \Gamma \mapsto \log \Psi[\Gamma] \) is a contraction, implying existence and uniqueness of the equilibrium inclusive value function \( \Gamma^* \). In the following, we let \( \|h\|_\infty \) denote the supremum norm \( \|h\|_\infty := \sup_{x \in X, z \in Z} |h(x, z)| \) of a function \( h(x, z) \).

**Theorem 3.1. (Existence and Uniqueness)** Under Assumption 2.1, (i) the mapping \( \log \Gamma \mapsto \log \Psi[\Gamma] \) is a contraction mapping,

\[
\|\log \Psi[\Gamma] - \log \Psi[\Gamma]\|_\infty \leq \lambda \|\log \Gamma - \log \hat{\Gamma}\|_\infty
\]

for any pair of functions \( \Gamma, \hat{\Gamma} \), where \( \lambda := \frac{\exp((\hat{U} + V) + \gamma^*)}{\Gamma + \exp(\hat{U} + V + \gamma^*)} < 1 \). Specifically, a solution to the fixed point problem in (3.5) exists and is unique. (ii) Moreover the equilibrium distributions are characterized by functions \( \Gamma^*_w(x) \) and \( \Gamma^*_m(z) \) that are continuous and \( p \) times differentiable in \( x \) and \( z \), respectively, with bounded partial derivatives.

See the appendix for a proof. The second part of the theorem is a straightforward implication of the observation that the respective ranges of the operators \( \Psi_w \) and \( \Psi_m \) are classes of functions with the same smoothness properties as the payoff functions \( U(x, z) \) and \( V(z, x) \).

It is important to note that we do not claim uniqueness of the stable matching in the limit, but only that the observable features of stable matchings converge to a unique limit. In fact, uniqueness of the limiting inclusive value functions can be easily reconciled with a growing number of distinct stable matchings as the number of agents in the market increases: If the stable matching is not unique, then by standard results from the theory of two-sided matching markets with non-transferable utilities, the number of available men in the women-preferred matching has to be weakly larger than in the men-preferred matching for all women, and strictly larger for at least one woman. We do in fact find in simulations that the number of individuals for whom the opportunity sets differ between the extremal matchings diverges to infinity as the market grows. However that difference grows more slowly than the typical size of the opportunity sets under any matching, so that under the scale normalization in the definition of \( \hat{\Gamma}_w \) and \( \hat{\Gamma}_m \), the gap between the inclusive values corresponding to the M-preferred and W-preferred matching vanishes as the number of individuals in the economy grows large.

### 3.6. Convergence of \( \hat{\Gamma} \) to \( \Gamma^* \).

We can now show that the solutions to the fixed-point problem in a finite economy (3.2) converge to the (unique) fixed point of the limiting problem (3.5): For one, the fixed-point mapping \( \hat{\Psi}_n \) is a sample average over functions of \( x, z \) and \( \Gamma \), and can be shown to converge in probability to its population expectation \( \Psi \), which defines
the limiting fixed point problem in (3.5). We then use this result to show that if an inclusive value function results from a stable matching, it can be represented as an approximate solution to the fixed point problem. Since the approximation is only shown to be valid for opportunity sets satisfying pairwise stability, the converse need not hold. However, the solution to the fixed point problem was shown to be unique in the previous section, and a stable matching is always guaranteed to exist, so that in fact the fixed point representation and pairwise stability are asymptotically equivalent.

Finally, since \( \log \Psi[\cdot] \) was shown to be a contraction, the solution to (3.5) is unique and well-separated in the sense that large perturbations of \( \Gamma \) relative to the fixed point \( \Gamma^* \) also lead to sufficiently large changes in \( \Gamma - \Psi[\Gamma] \). In particular, the functions \( \hat{\Gamma}_w^M, \hat{\Gamma}_m^M \) and \( \hat{\Gamma}_w^W, \hat{\Gamma}_m^W \) corresponding to the extremal matchings are solutions to (3.5) and therefore coincide even in the finite economy. Since the average inclusive value functions from the extremal matchings bound \( \hat{\Gamma}_w^*, \hat{\Gamma}_m^* \) for any other stable matching, the functions \( \hat{\Gamma}_w^*(x), \hat{\Gamma}_m^*(z) \) are uniquely determined.

Formally, we have the following theorem, which is proven in the appendix:

\textbf{Theorem 3.2.} Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then (a) for any stable matching, the inclusive values satisfy the fixed-point characterization in equation (3.2), and (b) we have convergence of the inclusive value functions \( \|\hat{\Gamma}_w^* - \Gamma_w^*\|_\infty \xrightarrow{p} 0 \) and \( \|\hat{\Gamma}_m^* - \Gamma_m^*\|_\infty \xrightarrow{p} 0 \).

The preceding steps established that for a large number of market participants, we can find an approximate parametrization of the distribution of matched characteristics with the inclusive value functions \( \Gamma_w \) and \( \Gamma_m \), which are characterized as solutions of a system of equilibrium conditions. The second part of Theorem 3.2 implies that the solutions of the sample equilibrium conditions converge to those of the limiting game discussed in the previous section. A converse of part (a) is not needed for our arguments, but with little additional work, it can be shown that it follows from existence of stable matchings in the finite economy together with uniqueness of the limit values of \( \Gamma \) established in Theorem 3.1.

3.7. \textbf{Convergence of Link Frequency Distributions.} We now turn to a characterization of the matching outcome in terms of the measure \( F(x, z) \) of matched characteristics corresponding to the number/mass of couples with observable characteristics \( x, z \) resulting from the matching. Given the inclusive value functions \( \Gamma_w^*(x) \) and \( \Gamma_m^*(z) \) characterizing the
equilibrium opportunity sets, we obtain the (mass) distribution
\[
f(x, z) = \frac{\exp\{U(x, z) + V(z, x) + \gamma_w + \gamma_m\}w(x)m(z)}{(1 + \Gamma^*_w(x))(1 + \Gamma^*_m(z))} \quad x \in \mathcal{X}, z \in \mathcal{Z}
\]
\[
f(x, \ast) = \frac{w(x)\exp\{\gamma_w\}}{1 + \Gamma^*_w(x)} \quad x \in \mathcal{X}
\]
\[
f(\ast, z) = \frac{m(z)\exp\{\gamma_m\}}{1 + \Gamma^*_m(z)} \quad z \in \mathcal{Z}
\]
where \(f(x, \ast)\) and \(f(\ast, z)\) denote the limiting densities of unmatched women of type \(x\), and unmatched men of type \(z\), respectively, and \(\Gamma^*_w\) and \(\Gamma^*_m\) solve the fixed-point equation in 3.5.

Since the inclusive value functions are asymptotically sufficient for describing the distribution of matched characteristics, convergence of \(\hat{\Gamma}\) to \(\Gamma_0\) also implies convergence of the matching frequencies in the finite economy to the (unique) limit \(f(x, z)\). Formally, we can now combine the conclusion of Theorem 3.2 with Lemma B.5 to obtain the limiting distribution:

**Corollary 3.1.** Under the assumptions of Theorem 3.2, for any sequence of stable matchings \(\mu^*_n\), the empirical matching frequencies \(\hat{F}_n(x, z|\mu^*_n)\) converge to a measure \(F(x, z)\) with density \(f(x, z)\) given in equation (3.6).

Broadly speaking, this result states that the matched characteristics from any stable matching converge “in distribution” to the limiting model in (3.6), where the measure \(F(x, z)\) and the empirical matching frequencies have the properties of proper probability distributions except that their overall mass is not normalized to one. Note that by our discussion in section 2.4 we can also obtain the sampling distribution for matched pairs under a given sampling protocol using the matching density \(f(x, z)\). This corollary is the main practical implication of our asymptotic analysis, and in the remainder of the paper we discuss some implications for identification and estimation of preference parameters from the distribution of matched characteristics.

To summarize, this limit has four important qualitative features: (1) all agents can choose among a large number of matching opportunities as the market grows, where (2) similar agents face similar options in terms of the inclusive value of their choice sets. Nevertheless, (3) the asymptotic sequence presumes that the option of remaining single remains sufficiently attractive relative to that rich set of potential matches. Finally (4) the number of distinct matchings may in fact be very large, but from the outsider’s perspective the different matchings become observationally equivalent in the limit. We find that these features are consistent with the idea of a matching market becoming “thick” as the number of participants becomes large.
4. Identification and Estimation

Since the main objective of this paper is to find estimators or tests that are consistent as the number of agents in the market increases, we analyze parameter identification from the limiting distribution of the matching game rather than its finite-market version. Since our identification results are based on likelihood ratios rather than absolute levels, we discuss identification based on direct knowledge of the measure $f(x, z)$ rather than the implied sampling distributions $h_1(x, z)$ or $h_2(x, z)$.

We first show how to transform the equilibrium matching probabilities derived in section 3 into a demand system that allows us to analyze identification in a fairly straightforward manner. Recall that for the limiting game, the joint density of observable characteristics for a matched pair was given in (3.6) by

$$f(x, z) = \frac{\exp\{U(x, z) + V(z, x) + \gamma_w + \gamma_m\} w(x) m(z)}{(1 + \Gamma_w^*(x))(1 + \Gamma_m^*(z))} \quad x \in \mathcal{X}, z \in \mathcal{Z}$$

where $\Gamma_w^*$ and $\Gamma_m^*$ solve the fixed point problem

$$\Gamma_w^*(x) := \int \frac{\exp\{U(x, s) + V(s, x) + \gamma_m\} m(s)}{1 + \Gamma_m^*(s)} ds$$

$$\Gamma_m^*(z) := \int \frac{\exp\{U(s, z) + V(z, s) + \gamma_w\} w(s)}{1 + \Gamma_w^*(s)} ds$$

We also define the marital pseudo-surplus of a match as the sum of the deterministic parts of random payoffs,

$$W(x, z) := U(x, z) + V(z, x)$$

where in the absence of a common numeraire, the relative scales of men and women’s preference are normalized by a multiple of the conditional standard deviation of random utilities given $x_i = x$ and $z_j = z$. In order to understand identifying properties of the model, it is important to note that we can express the distribution in terms of $W(x, z)$ alone:

$$f(x, z) = \frac{\exp\{W(x, z) + \gamma_w + \gamma_m\} w(x) m(z)}{(1 + \Gamma_w^*(x))(1 + \Gamma_m^*(z))} \quad x \in \mathcal{X}, z \in \mathcal{Z} \quad (4.1)$$

Specifically, our proofs of large-sample results for estimators are analogous to generic convergence arguments for extremum estimation (see e.g. Newey and McFadden (1994) sections 2 and 3, or van der Vaart and Wellner (1996) chapter 3.2-3), where the population parameter of interest is the unique maximizer of the (limiting) population objective function, which is then approximated by sample quantities. These arguments do not require that the sampling objective function results from the same data generating process (DGP) as the limiting objective as long as we have uniform convergence along the triangular sequence of DGPs.
Also, the fixed-point equations defining $\Gamma_w^*$ and $\Gamma_m^*$ can be rewritten as

$$
\Gamma_w^*(x) := \int \frac{\exp\{W(x, s) + \gamma_m\} m(s)}{1 + \Gamma_m^*(s)} ds
$$

$$
\Gamma_m^*(z) := \int \frac{\exp\{W(s, z) + \gamma_w\} w(s)}{1 + \Gamma_w^*(s)} ds
$$

(4.2)

Hence, the joint distribution of matching characteristics depends on the systematic parts of $U_{ij}$ and $V_{ji}$ only through $W(x, z)$, so that we can in general not identify $U(x, z)$ and $V(z, x)$ separately without additional restrictions. This naturally limits the possibilities for welfare assessments from the perspective of either side of the market, but conversely, the pseudo-surplus $W(x, z)$ is, at least in the limit, a sufficient statistic for the impact of observable characteristics on the resulting stable matching.

4.1. Identification. We next consider identification of the pseudo-surplus function $W(x, z)$:

Taking logs on both sides and rearranging terms, we obtain

$$
\log f(x, z) - [\log w(x) + \log m(z)] = W(x, z) - [\log(1 + \Gamma_w^*(x)) + \log(1 + \Gamma_m^*(z))] \quad (4.3)
$$

Note that the terms on the left-hand side are features of the distribution of observable characteristics and can be estimated from the data. This expression also suggests a computational shortcut which obviates the need to infer the (unobserved) opportunity sets $m(z, x)$ and $w(x, z)$ to compute the matching probabilities. We can use differencing arguments to eliminate the inclusive values $\Gamma_w^*(x)$ and $\Gamma_m^*(z)$ from the right-hand side expression in (4.3) using information on the shares of unmatched individuals. Specifically, for any values $x \in X$ and $z \in Z$, consider differences of the form

$$
\log \frac{f(x, z)}{f(x, *) f(*)} = W(x, z)
$$

Since the quantities on the right-hand side are observed, the joint distribution of matched characteristics identifies the pseudo-surplus function $W(x, z)$. This means we can directly estimate the strength of complementarities between observable types from differences in logs of the p.d.f. of characteristics in matched pairs.

This simplification can be seen as a direct consequence of the IIA assumption, where dependence of choice probabilities on a latent opportunity set is entirely captured by the inclusive value. Similar differencing arguments have been used widely in the context of discrete choice models for product demand, see Berry (1994).

Furthermore, we can identify the average inclusive value function from the conditional probabilities of remaining single given different values of $x$,

$$
\Gamma_w(x) = \frac{w(x) \exp\{\gamma_w\}}{f(x, *)} - 1
$$
We can summarize these findings in the following proposition:

**Proposition 4.1. (Identification)** (a) The surplus function $W(x, z)$ and the inclusive value functions $\Gamma_w(x)$ and $\Gamma_m(z)$ are point-identified from the limiting measure $f(x, z)$. (b) Without further restrictions, the systematic parts of the random utilities, $U(x, z)$ and $V(z, x)$ are not separately identified from the limiting measure $f(x, z)$.

While non-identification of the random utility functions $U(x, z)$ and $V(z, x)$ is a negative result, it is important to note that the surplus function $W(x, z) := U(x, z) + V(z, x)$ is an object of interest in itself. Most importantly, the characterization of the limiting distribution implies that knowledge of $W(x, z)$ is sufficient to compute any counterfactual distributions of match characteristics and analyze welfare consequences of policy changes. E.g. it is possible to predict the effect of changes in sex ratios on marriage rates based on the surplus function $W(x, z)$ alone.

However, in the context of a parametric model for $U(x, z; \theta)$ and $V(z, x; \theta)$ it may be possible to identify preference parameters for the two sides of the market separately in the presence of exclusion restrictions. Since in a two-sided matching market “demand” of one side constitutes “supply” on the other, this identification problem illustrates the close parallels with estimation of supply and demand functions from market outcomes. In that case, our asymptotic characterization of matching probabilities allows to formulate rank conditions for identification directly in terms of the surplus function $W(x, z)$. As an example, in mechanisms for assigning students to schools, public schools typically prioritize students based on very few characteristics - e.g. applicants who live within walking distance, or whose siblings attend the same institution - and ignore other variables. While preferences of students’ families likely depend on those characteristics as well, it would still be possible to identify the incremental effect of any other student-specific variables.\(^{10}\)

Comparing this result to the previous literature on identification in matching markets, it can be seen that the decision whether to model idiosyncratic preferences over types or individuals has crucial implications as to whether we can only identify marital surplus or individual utilities. Our identification result suggests that for “thick” markets, models with and without transferable utilities may lead to similar qualitative implications, however in the latter case information on actual transfers may be useful for identification.

4.2. **Estimation.** We now turn to estimation of a parametric model for the systematic payoffs $U^*(x, z) := U^*(x, z; \theta)$ and $V^*(z, x) := V^*(z, x; \theta)$, where the unknown parameter $\theta \in \Theta_0$, a compact subset of some Euclidean space. We can then estimate the parameter $\theta$ up to

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\(^{10}\)However we should point out that since the school mechanism is typically a “many to one” assignment problem, our asymptotic results would strictly speaking only apply in the unrealistic case in which students have distinct preferences over “seats” in each school, and priorities on the school side of the mechanism are randomized independently across schools and across “seats.”
the normalizations necessary for identification using the resulting surplus function \( W(x, z) = W(x, z; \theta) := U^*(x, z; \theta) + V^*(z, x; \theta) \) Without loss of generality, assume that estimation is based on a random sample of individuals rather than couples, with sampling distribution given in (2.3). Considering log-ratios of the density of match characteristics is convenient for the analysis of identification of the model without explicitly characterizing the (unobserved) set of available spouses for any individual. However, two-step estimation of payoff parameters based on first-step estimates of the p.d.f. of the sampling distribution \( h_1(x, z) \) is impractical for estimation, since fully nonparametric estimation of the density would suffer from a curse of dimensionality in most realistic settings. On the other hand, imposing the restrictions resulting from our knowledge on the function \( W(x, z) \) when estimating \( h_1(x, z) \) requires solving for the equilibrium distribution and latent inclusive values, which we wanted to avoid in the first place.

Instead we suggest to treat the inclusive values \( \Gamma_{ww(k)} := \Gamma_w(x_{w(k)}), \Gamma_{mm(k)} := \Gamma_m(z_{m(k)}) \) as auxiliary parameters in maximum likelihood estimation of parameters of the surplus function \( W(x, z) \), and imposing equilibrium conditions as side constraints in a joint maximization problem over \( \theta \) and \( \Gamma_{ww(k)}, \Gamma_{mm(k)} \), where \( k = 1, \ldots, K \). Specifically, it follows from equation (2.3) that

\[
\begin{align*}
 l_K(\theta, \Gamma) := \log h_1(x_{w(k)}, z_{m(k)}|\theta, \Gamma) \\
= W(x_{w(k)}, z_{m(k)}; \theta) + \log(2) \mathbb{1}\{w(k) \neq 0, m(k) \neq 0\} \\
- \log(1 + \Gamma_w(x_{w(k)})) - \log(1 + \Gamma_m(z_{m(k)})) + \text{const}
\end{align*}
\]

Hence we can write the likelihood function (up to a constant) as

\[
L_K(\theta, \Gamma) := \sum_{k=1}^{K} \left( W(x_{w(k)}, z_{m(k)}; \theta) + \log(2) \mathbb{1}\{w(k) \neq 0, m(k) \neq 0\} \\
- \log(1 + \Gamma_w(x_{w(k)})) - \log(1 + \Gamma_m(z_{m(k)})) \right)
\]

We can also state the equilibrium conditions using the operator \( \hat{\Psi}_K := (\hat{\Psi}_wK, \hat{\Psi}_mK) \) where

\[
\hat{\Psi}_wK[\Gamma](x) = \frac{1}{K} \sum_{k=1}^{K} \exp \left\{ W(x, z_{m(k)}; \theta) \right\} \mathbb{1}\{m(k) \neq 0\} \\
\hat{\Psi}_mK[\Gamma](z) = \frac{1}{K} \sum_{k=1}^{K} \exp \left\{ W(x_{w(k)}, z; \theta) \right\} \mathbb{1}\{w(k) \neq 0\}
\]

Hence the maximum likelihood estimator \( \hat{\theta} \) solves

\[
\max_{\theta, \Gamma} L_K(\theta, \Gamma) \quad \text{s.t.} \Gamma = \hat{\Psi}_K(\Gamma)
\]
We can find the MLE $\hat{\theta}$ by constrained maximization of the log-likelihood, where treating the equilibrium conditions as constraints obviates the need of solving for an equilibrium in $\Gamma$ at every maximization step. This constrained problem is high-dimensional in that $\Gamma$ generally has to be evaluated at up to $2^K$ different arguments, leading to $2^K$ constraints. That system of constraints is not sparse but collinear, and we may adapt the constrained MPEC algorithm proposed and described in Dubé, Fox, and Su (2012) and Su and Judd (2012).

Since the equilibrium conditions for the inclusive value functions are market-specific, consistent estimation will generally require that we observe a large number of individuals or couples in each market. However in the absence of market-level heterogeneity, data from any finite number of markets is typically sufficient for consistency since all objects of interest are identified from the distribution in a single market. Asymptotic inference for surplus parameters or counterfactuals is standard if the sample used by the researcher contains a small fraction of the households or individuals in each market, and is drawn at random from the relevant population. If the researcher’s sample includes the entire market, or a significant share of all individuals, then structural inference on surplus parameters has to account for conditional dependence of matching decisions across pairs. Asymptotics of this type have been developed in Menzel (2012) for discrete games, however an application of these ideas to matching markets is not straightforward and will be left for future research.

5. Extensions: Limited Awareness, Returns to Scale, and Welfare

The baseline model from the previous section can be extended or interpreted in various additional ways that are important for empirical work. For one, we relax the assumption of agents being aware of all other market participants by introducing a static search friction into the model. We then show how the matching function for the generalized model scales with the market size as measured by the parameters $\gamma_w, \gamma_m$, and give a simple condition under which returns to scale are constant. Finally, we discuss how to conduct welfare analyses based on the matching model.

5.1. Matching with Limited Awareness. The asymptotic results on uniqueness of the limiting distribution and convergence in the previous sections require that the outside option remain relevant even as the number of available partners grows for each agent in the market. The asymptotic sequence of markets in our limiting experiment involved drawing an increasing number of random utilities for the outside option, and was motivated by approximating properties. A more plausible behavioral explanation why many agents remain unmatched even in large markets is that with a growing size of the market, each agent may not be aware of all her/his matching opportunities. Our asymptotic results require that the
number of available matches grows for each agent, most importantly the step in Lemma B.5, and cannot be easily adjusted to a setup in which opportunity sets remain small.

This section develops an extension of our baseline setup in which every agent only becomes aware of a subset of potential matches at random, where the probability of meeting a potential spouse may be a function of observed characteristics. For example, characteristics may include geographic location, so that the probability of meeting may be a function of spatial distance. This modified thought experiment maintains that the number of draws for the outside option grow at a root-n rate with the size of the market and continues to assume that opportunity sets be large. The main purpose of this extension is to arrive at a more realistic interpretation of the pseudo-surplus function $W^*(x, z)$ in a setting in which agents may only observe a subset of agents in the market.

In the modified model, nature initially draws random utilities $U_{ij}, V_{ji}$ according to the model in (2.1) and (2.2). The matching market then operates in two stages: in stage 1, agents meet at random and independently of the realized random matching payoffs (“interview”), where the probability of a woman of type $x$ meeting a man of type $z$ is given by $r(x, z) \in [0, 1]$. Awareness is assumed to be mutual, i.e. woman $i$ is aware of man $j$ if and only if man $j$ is also aware of woman $i$. If a pair $(i, j)$ of a woman and a man is not aware of each other, the random payoffs for a match between them are set to minus infinity, and otherwise equal to their initial values, i.e.

$$
\tilde{U}_{ij,n} = \begin{cases} 
U_{ij,n} & \text{if } i \text{ and } j \text{ meet} \\
-\infty & \text{otherwise}
\end{cases}
$$

$$
\tilde{V}_{ji,n} = \begin{cases} 
V_{ji,n} & \text{if } i \text{ and } j \text{ meet} \\
-\infty & \text{otherwise}
\end{cases}
$$

In stage 2, the market mechanism determines a matching that is stable with respect to the modified payoffs $\tilde{U}_{ij,n}, \tilde{V}_{ji,n}$, and which is observed by the researcher. Note that in the presence of an outside option of remaining single, the modified payoffs continue to satisfy the assumptions of Gale and Shapley (1962)’s model of stable marriage. In particular, our model of matching with limited awareness is guaranteed to produce at least one stable matching, and the set of stable matchings is a lattice under the preference orderings of the male and female side of the market, respectively. This two-stage process is very similar to the interviewing game in Lee and Schwarz (2012), except that we do not solve explicitly for an equilibrium in the number of interviews that each party chooses to conduct.

It is then straightforward to verify that the argument leading to Corollary 3.1 can be adjusted to accommodate this extension, and we obtain the following limiting result:

**Proposition 5.1.** Suppose Assumptions 2.1, 2.2, and 2.3 hold, and furthermore $|\log r(x, z)| \leq \bar{R}$, a finite constant, for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$. Then the distribution of matched characteristics is characterized by equations (4.1) and (4.2), where

$$W^*(x, z) := U^*(x, z) + V^*(x, z) + \log r(x, z)$$
In particular, \( W^*(x, z) \) is nonparametrically identified from the measure \( f(x, z) \).

We can now directly apply our results on identification and estimation of the pseudo-surplus function to the modified matching game. Most importantly, we can see from the form of the limiting distribution of matchings in Proposition 5.1 that for large markets we cannot separate whether observable characteristics affect the matching frequency through the likelihood of meeting or preferences.

5.2. Scaling Properties. It is furthermore interesting to analyze the returns to scale with respect to market size on the matching function implied by our limiting model. For their model of transferable utilities with finite types, Choo and Siow (2006) show that the matching function is homogeneous of degree zero, so that in particular the rate of singles in the population does not depend on the number of market participants. However, our setup differs qualitatively from theirs in that we assume that idiosyncratic taste-shifters are individual-rather than type-specific, so that a larger number of matching opportunities increases the attractiveness of the best available partner relative to the outside option. Furthermore the limiting matching function in (3.6) corresponding to the nontransferable utility model is different from their result.

While under our asymptotics, all markets are assumed to grow large, our setup still allows for differences in relative scale by varying the parameters \( \gamma_m, \gamma_w \). We can see that in general, the matching functions are not homogeneous of degree zero, but other things equal, increasing \( \gamma_m \) and \( \gamma_w \) simultaneously leads to a decrease in the share of singles. However, in the model with limited awareness we can eliminate scale effects by assuming that \( \gamma_w + \log r(x, z) \) and \( \gamma_m + \log r(x, z) \) are constant across markets.\(^{11}\) We can interpret this comparison as individuals in each market being aware of roughly the same number of potential spouses regardless of market size.

Finally, we can analyze the effect of an increase of \( \gamma_w \) that leaves \( \gamma_m \) constant. From the fixed-point conditions (3.4) we can see that such a change would result in an increase of inclusive values for men and a decrease in inclusive values for women. From (3.6) we can then conclude that as a result, the share of singles among men would decrease, whereas the share of singles among women would increase. From an analogous argument, altering the marginal distributions \( w(x) \) and \( m(z) \) would result in similar effects on those types of men or women that become more “scarce” or “abundant,” respectively.

5.3. Welfare Effects. In addition to prediction of counterfactuals regarding observable characteristics of the realized matches, our model also allows for welfare evaluations of policy interventions. Recall that the inclusive value is related to the expectation of indirect utility

\(^{11}\)Note that under this restriction, the fixed-point conditions in (3.5) are solved by the same values for \( \Gamma_w \) and \( \Gamma_m \) for each market, so that the share of singles implied by (3.6) remains the same.
via
\[
\mathbb{E}[U^*_i|X_i = x] = \log(1 + \Gamma_w(x)) + \kappa
\]
where \(\kappa \approx 0.5772\) is Euler’s constant. In particular, if we define \(s_w(x) := \frac{f(x,*)}{w(x)\exp(\gamma_w)}\) as the share of women of type \(x\) that remain single, we can express the inclusive value in terms of observable quantities,
\[
\mathbb{E}[U^*_i|X_i = x] = -\log \frac{f(x,*)}{w(x)} + \gamma_w + \kappa = -\log s_w(x) + \text{const}
\]
Hence, for natural experiments that change the composition of matching markets, we can interpret the difference in the log shares of unmatched individuals directly as the average change in the surplus from participating in the matching market for individuals of a given observable type.

For example we can evaluate the effect of changes to the marginal distribution of characteristics in the market, \(w(x)\) and \(m(z)\) on individual surplus for any type on either side of the market. If observed characteristics include income and own education, we can identify the monetary return to education on the marriage market (compensating or equivalent variation) from local shifts in education levels and incomes that leave the share of unmatched individuals constant.\(^{12}\) Clearly, changes in individuals’ types also affect the value of their outside option - which was normalized to zero in our analysis - so that comparisons based log \(s_w(x)\) do not capture welfare changes that do not operate through the matching market.

For example, we may observe that an increase in women’s education leads to an increase in the share of unmarried women. This may either reflect a deterioration of women’s matching prospects, or an increase in the relative value of their outside option, either of which is associated with a decrease in her expected (net) return from participating in the matching market.

6. Monte Carlo Simulations

In order to illustrate the different aspects of the theoretical convergence result, we simulate a very basic version of our model. We generate payoff matrices from the random utility model
\[
U_{ij} = U^*(x_i, z_j) + \sigma_n \eta_{ij}
\]
\[
V_{ji} = V^*(z_j, x_i) + \sigma_n \zeta_{ji}
\]
where idiosyncratic taste shifters \(\eta_{ij}, \zeta_{ji}\) are i.i.d. draws from a standard normal or extreme value type-I distribution. We then find the \(M\)- and \(W\)-preferred matchings using the Gale-Shapley (deferred acceptance) algorithm.

\(^{12}\)see Small and Rosen (1981)
6.1. Approximation of Matching Probabilities. We first illustrate convergence of matching frequencies generated by the model to the limiting choice probabilities predicted by the asymptotic arguments in the previous sections. Specifically, we illustrate three qualitative conclusions of our theoretical results: (1) the degree of multiplicity of matching outcomes increases in the size of the market, but (2) that growth is not fast enough to affect the limiting distribution of matched characteristics. (3) Convergence of conditional choice probabilities to their extreme value limits when the respective distribution of \( \eta_{ij} \) and \( \zeta_{ji} \) are not extreme-value type I can very slow in some cases, most importantly when taste shifters are generated from the standard normal distribution.

Our first set of simulation results is based on a design with taste-shifters \( \eta_{ij}, \zeta_{ji} \) generated from the extreme-value type I distribution and no observable characteristics. In table 1 we report the difference in the average size of a woman’s opportunity set between the extremal matchings, \(|M^W_i| - |M^M_i|\), where \( M^W_i \) denotes woman \( i \)'s opportunity set under the female-preferred matching, and \( M^M_i \) her opportunity set under the male-preferred matching. As argued before, opportunity sets arising from stable matchings are nested: For any given stable matching \( \mu^* \), woman \( i \)'s opportunity set under \( \mu^* \) satisfies \( M^M_i \subset M_i[\mu^*] \subset M^W_i \) with probability 1. Hence we can interpret the difference between the two extremal matchings as an upper bound on the variation of opportunity sets across different stable matchings. We also report the number of women for whom \( M_i \) differs across matchings, i.e. \( M^W_i \not\subset M^M_i \) and the average inclusive values for the \( W^- \) and \( M^- \) preferred matchings.

| n   | \(|M^W_i| - |M^M_i|\) | \#\{\(M^W_i \not\subset M^M_i\)\} | \(I_w[M^W_i]\) | \(I_w[M^M_i] - I_w[M^M_i]\) |
|-----|------------------|---------------------------------|-------------|------------------|
| 10  | 0.0350           | 0.30                            | 6.73        | 0.08             |
| 20  | 0.2125           | 3.05                            | 7.50        | 0.35             |
| 50  | 0.1370           | 5.90                            | 6.92        | 0.14             |
| 100 | 0.0905           | 8.35                            | 6.77        | 0.07             |
| 200 | 0.1175           | 21.95                           | 6.82        | 0.06             |
| 500 | 0.0574           | 27.50                           | 6.87        | 0.02             |
| 1000| 0.0539           | 52.00                           | 6.85        | 0.01             |
| 2000| 0.0510           | 92.60                           | 6.86        | 0.01             |
| 5000| 0.0041           | 102.00                          | 6.87        | 0.00             |

Table 1. Comparisons of the Male and Female Preferred Matchings (Extreme-Value Type I Taste Shifters).

From table 1, we can see that as \( n \) grows, the number of women for whom there is a difference in opportunity sets across stable matchings increases steadily, although at a rate that is less than proportional to \( n \). Furthermore, the average difference in the number of available spouses decreases in \( n \), as does the difference in inclusive values. For the latter, this is in part due to the normalization of the inclusive values which are scaled by the inverse of
root-n. Given that normalization, inclusive values converge to a nonzero limit, and the simulations also show that the variance of $I_w[M^*_i]$ (not reported in the table) decreases to zero. These simulation results suggest that realizations of payoffs that support an exponentially increasing number of stable matchings as in Theorem 3.19 in Roth and Sotomayor (1990) are not “typical” for the random utility model analyzed in this paper.

To verify the quality of the approximation to the distribution of matched types, we also compare the probability of remaining single predicted by the model and the simulated frequency of singles in the $W$–preferred stable matching. In this setting, the matching probabilities are the same under the $W$– and $M$–preferred matching by the “Rural Hospital Theorem.” The simulation results are reported in table 2. We can see that the quality of

<table>
<thead>
<tr>
<th>n</th>
<th>Model Probability</th>
<th>Simulated Frequency</th>
<th>Bias (2 minus 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.1265</td>
<td>0.1760</td>
<td>-0.0495</td>
</tr>
<tr>
<td>20</td>
<td>0.1265</td>
<td>0.1510</td>
<td>-0.0245</td>
</tr>
<tr>
<td>50</td>
<td>0.1265</td>
<td>0.1392</td>
<td>-0.0127</td>
</tr>
<tr>
<td>100</td>
<td>0.1265</td>
<td>0.1358</td>
<td>-0.0093</td>
</tr>
<tr>
<td>200</td>
<td>0.1265</td>
<td>0.1322</td>
<td>-0.0057</td>
</tr>
<tr>
<td>500</td>
<td>0.1265</td>
<td>0.1333</td>
<td>-0.0068</td>
</tr>
<tr>
<td>1000</td>
<td>0.1265</td>
<td>0.1338</td>
<td>-0.0074</td>
</tr>
<tr>
<td>2000</td>
<td>0.1265</td>
<td>0.1300</td>
<td>-0.0035</td>
</tr>
<tr>
<td>5000</td>
<td>0.1265</td>
<td>0.1270</td>
<td>-0.0005</td>
</tr>
</tbody>
</table>

Table 2. Theoretical and Simulated Matching Frequencies (Extreme-Value Type I Taste Shifters).

the approximation improves as $n$ grows large, however markets have to be quite large for the approximation bias to be small, say less than half a percentage point. The asymptotic arguments suggest that the convergence rate for matching probabilities should be $n^{-1/4}$, regardless whether observed covariates are continuous or discrete.

Next, we repeat the same experiment with standard normal taste shifters to evaluate the quality of the extreme-value approximation for the conditional choice probabilities. In general, convergence rates for distributions of extremes depend on the shape of the tails of the c.d.f. of random taste shifters, and especially for a thin-tailed distribution the convergence rate is very slow.\(^{13}\) Hence we should not expect the approximation to improve very much for moderate sample sizes. Table 3 reports results on differences in inclusive values and opportunity sets across stable matchings, and it is interesting to see that the qualitative results on multiplicity of stable matchings remain unchanged as we alter the distribution

\(^{13}\)Hall (1979) showed that the rate of convergence for the c.d.f. of the maximum of independent normals is $\log n$. 
of unobservables. Most importantly, the number of agents for whom there is a difference between the extremal matchings still grows in \( n \), whereas the inclusive values converge to a common limit, independently of the chosen stable matching. We also compare the predicted

| \( n \) | \( |M_i^W| - |M_i^M| \) | \( \#\{M_i^W \neq M_i^M\} \) | \( I_w[M_i^W] \) | \( I_w[M_i^M] - I_w[M_i^M] \) |
|---|---|---|---|---|
| 10 | 0.2340 | 1.86 | 7.37 | 0.55 |
| 20 | 0.2820 | 3.86 | 7.30 | 0.47 |
| 50 | 0.1920 | 7.90 | 7.20 | 0.20 |
| 100 | 0.1476 | 11.54 | 7.13 | 0.11 |
| 200 | 0.1196 | 21.18 | 7.21 | 0.06 |
| 500 | 0.0583 | 26.10 | 7.08 | 0.02 |
| 1000 | 0.0461 | 44.18 | 6.98 | 0.01 |

**Table 3.** Comparisons of the Male and Female Preferred Matchings (Standard Normal Taste Shifters).

(asymptotic) probability of remaining single with the simulated frequencies in table 4. Here we can see that even for large markets, matching frequency approach the theoretical limit only up to a point, and then convergence becomes very slow as should be expected in light of the slow convergence rates of normal extremes. For realistic market sizes it would therefore be more plausible to impose extreme-value type I taste shifters as an assumption rather than arising from a many-alternative limit. However, the asymptotic results on conditional choice probabilities still imply that the conditional Logit specification is the only asymptotically stable model for taste shifters that have distributions with tails of type I.

6.2. **ML Estimation of Preference Parameters.** Finally we give some Mote Carlo evidence on estimation of structural preference parameters from the realized matching. Our
simulation design specifies systematic utilities

\[ U^*(x_i, z_j) = \theta_0 + \theta_1 x_i + \theta_2 x_i z_j \]

\[ V^*(z_j, x_i) = \theta_0 + \theta_1 z_j + \theta_2 x_i z_j \]

for scalar parameters \( \theta_0, \theta_1, \theta_2 \), where types for men and women, \( x_i, z_j \in \{0, 1\} \) are generated from a symmetric Bernoulli distribution, and idiosyncratic taste shifters \( \eta_{ij}, \zeta_{ji} \) are i.i.d. draws from the extreme-value type I distribution. The parameter \( \theta_1 \) measures the systematic difference in the propensity for marriage between the two types, and \( \theta_2 \neq 0 \) generates matchings that are assortative across observable type categories.

In each Monte Carlo replication, we simulate data for one single market of size \( n \) via the Gale-Shapley algorithm, assuming the W-optimal matching. Since with discrete types, the limiting estimation problem is finite-dimensional, we used the \texttt{fmincon} command in Matlab to obtain the constrained maximum likelihood estimators.\(^\text{14}\) For any given simulated data set, the computational cost of finding the MLE is fairly small, however computation of the extremal matchings using the deferred acceptance algorithm becomes very costly as \( n \) increases. We therefore simulate markets of sizes only up to \( n = 2000 \). Table 5 reports the simulated mean and standard deviation of the distribution of the estimators based on 200 independent Monte Carlo replications. We discarded draws of markets for which the standard minimization routine for computing the MLE did not achieve convergence.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \theta_0 )</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.7557 (0.4491)</td>
<td>0.3598 (0.6762)</td>
<td>0.7236 (0.1845)</td>
</tr>
<tr>
<td>50</td>
<td>0.4634 (0.3968)</td>
<td>0.4523 (0.5545)</td>
<td>0.9334 (0.2138)</td>
</tr>
<tr>
<td>100</td>
<td>0.4162 (0.3285)</td>
<td>0.5386 (0.4480)</td>
<td>0.9716 (0.1868)</td>
</tr>
<tr>
<td>200</td>
<td>0.4512 (0.2549)</td>
<td>0.5034 (0.3297)</td>
<td>0.9702 (0.1378)</td>
</tr>
<tr>
<td>500</td>
<td>0.4892 (0.1973)</td>
<td>0.5103 (0.2622)</td>
<td>0.9784 (0.0833)</td>
</tr>
<tr>
<td>1000</td>
<td>0.4801 (0.1081)</td>
<td>0.4977 (0.1297)</td>
<td>0.9710 (0.0579)</td>
</tr>
<tr>
<td>2000</td>
<td>0.4795 (0.0693)</td>
<td>0.4931 (0.1005)</td>
<td>0.9941 (0.0400)</td>
</tr>
<tr>
<td>(DGP)</td>
<td>0.50</td>
<td>0.50</td>
<td>1.00</td>
</tr>
</tbody>
</table>

| TABLE 5. Distribution of Maximum Likelihood Estimators for Preference Parameters: mean and standard deviation (in parentheses) |

We can see that for matching markets of moderate to large sizes, the estimates become concentrated near the values specified in the data generating process, and the standard deviation of the estimator decreases as the market size increases. One important feature of the simulation results is that the bias of the MLE decreases fairly slowly as \( n \) grows and

\(^{14}\) For the components of \( \theta \), we used zeros as starting values, the components of \( \Gamma \) were initialized at one. Alternative starting values did not produce different results except for instances in which the minimization algorithm did not converge.
appears to be of a similar order of magnitude as the standard error. This is in contrast to the asymptotic behavior of regular nonlinear estimators with i.i.d. data (see e.g. Rilstone, Srivastava, and Ullah (1996)), where the only source of bias is sampling error. In the present setting, the MLE is also subject to approximation error from using the limiting model rather than the exact finite-player distribution, so that the bias of the estimator need not be negligible relative to its standard error even for very large values of $n$.

While we do not derive the convergence rate for the MLE, our simulation results are compatible with a parametric root-n rate for $\hat{\theta}_{ML}$. In order to illustrate the relative magnitudes of the different approximation errors, table 6 reports estimator bias and standard errors multiplied by $n^{1/2}$. We also compute “total” root mean-square error $RMSE = \left( n \sum_j E[(\hat{\theta}_j - \theta_j)^2] \right)^{1/2}$ as a summary measure of estimator precision, and report the maximum of studentized bias over components of $\hat{\theta}_j$, $SBR := \max_j |Bias(\hat{\theta}_j)|/SE(\hat{\theta}_j)$ to assess the relative magnitude of bias relative to estimator dispersion.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\theta_0$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>RMSE</th>
<th>SBR</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.1435</td>
<td>-0.6270</td>
<td>-1.2361</td>
<td>4.1338</td>
<td>1.4981</td>
</tr>
<tr>
<td>50</td>
<td>-0.2588</td>
<td>-0.3373</td>
<td>-0.4709</td>
<td>5.0926</td>
<td>0.3115</td>
</tr>
<tr>
<td>100</td>
<td>-0.8380</td>
<td>0.3860</td>
<td>-0.2840</td>
<td>5.9399</td>
<td>0.2551</td>
</tr>
<tr>
<td>200</td>
<td>-0.6901</td>
<td>0.0481</td>
<td>-0.4214</td>
<td>6.2601</td>
<td>0.2163</td>
</tr>
<tr>
<td>500</td>
<td>-0.2415</td>
<td>0.2303</td>
<td>-0.4830</td>
<td>7.5929</td>
<td>0.2593</td>
</tr>
<tr>
<td>1000</td>
<td>-0.6293</td>
<td>-0.0727</td>
<td>-0.9171</td>
<td>5.7535</td>
<td>0.5009</td>
</tr>
<tr>
<td>2000</td>
<td>-0.9168</td>
<td>-0.3086</td>
<td>-0.2639</td>
<td>5.8319</td>
<td>0.2958</td>
</tr>
</tbody>
</table>

**Table 6.** Scaled bias and standard deviation for the MLE (in parentheses), total RMSE, and studentized bias ratios.

We find that across the range of specifications in our simulation study, bias and standard error of the parameters are both roughly of the order $n^{-1/2}$, and the RMSE and SBR measures are of similar magnitude for all values of $n$. In particular, the SBR measure does not appear to converge to zero even for large values of $n$. This suggests that the asymptotic bias of the MLE is of first order if the total number of observations in the researcher’s sample is comparable to the size of markets from which those observations are drawn. In that case, correct inference based on the MLE would require the use of bias correction techniques, the development of which is beyond the scope of this paper. On the other hand, if the MLE is computed based on a data set that is small relative to typical market size, sampling uncertainty may dominate the approximation error so that distortions from using standard inference techniques would be negligible.
APPENDIX A. PROOF OF LEMMA 2.1

To verify that the statement in Lemma 2.1 is indeed equivalent to the usual definition of pairwise stability, notice that if \( \mu \) is not pairwise stable, there exists a pair \( i, j \neq \mu(i) \) such that \( U_{ij} > U_{ijw(i)} \) and \( V_{ji} > V_{j\mu_m(j)} \). In particular, \( j \) is available to \( i \) under \( \mu \), i.e. \( j \in M_i[\mu] \), so that \( U^*_i(M_i[\mu]) \geq U_{ij} > U_{ijw(i)} \) which violates the first part of the condition. Conversely, if the condition in the Lemma does not hold for woman \( i \), then there exists \( j \in W^*_i[\mu] \) such that \( U^*_i(M_i[\mu]) \geq U_{ij} > U_{ijw(i)} \). On the other hand, \( j \in M_i[\mu] \) implies that \( V_{ji} \geq V_{j\mu_m(j)} \), and that inequality is strict in the absence of ties since by assumption \( i \neq \mu_m(j) \). On the other hand, if the condition is violated for a man \( j \), we can find a blocking pair consisting of \( j \) and a woman \( i \in W_j[\mu] \) using an analogous argument. □

APPENDIX B. PROOFS FOR SECTION 3

B.1. Proof of Theorem 3.1: Without loss of generality, let \( \gamma^* = 0 \). The proof consists of three steps: We first show that under Assumption 2.1, any solution to the fixed point problem in (3.5) is differentiable, so that we can restrict the problem to fixed points in a Banach space of continuous functions. We then show that the mapping \((\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])\) is a contraction, so that the conclusions of the theorem follow from Banach’s fixed point theorem. Without loss of generality, we only consider the case in which all observable characteristics are continuously distributed, \( x_{1i} = x_i \) and \( z_{1i} = z_i \).

Bounds on solutions. We first establish that any pair of functions \((\Gamma^*_{w}(x), \Gamma^*_m(z))\) solving the fixed point problem in (3.5) are bounded from above: Assuming the solutions exist, and noticing that \( \Gamma^*_m(z) \geq 0 \) for all \( z \in Z \), we have that

\[
\Gamma^*_w(x) = \Psi_w[\Gamma^*_m](x) = \int \frac{\exp\{U(x, s) + V(s, x)\}m(s)}{1 + \Gamma^*_m(s)} ds 
\leq \int \exp\{U(x, s) + V(s, x)\}m(s) ds \leq \exp\{\bar{U} + \bar{V}\} \tag{B.1}
\]

which is finite by assumption 2.1. Similarly, we can see that

\[
\Gamma^*_m(z) \leq \exp\{\bar{U} + \bar{V}\} \tag{B.2}
\]

if a solution to the fixed point problem exists.

Continuity of solutions. In order to establish continuity, notice that any fixed point \((\Gamma_w, \Gamma_m)\) of \((\Psi_w, \Psi_m)\) has to satisfy

\[
\Gamma_w = \Psi_w[\Psi_m[\Gamma_w]]
\]

Now, consecutive application of \( \Psi_w \) and \( \Psi_m \) gives

\[
\Psi_w[\Psi_m[\Gamma_w]](x) = \int \frac{\exp\{U(x, t) + V(t, x)\}}{1 + \int \frac{\exp\{U(x, s) + V(s, z)\}m(z) ds}{1 + \Gamma_w(s)} m(t) dt}
\]

for any function \( \Gamma_w \). Since \( \exp\{U(x, z)\} \) and \( \exp\{V(z, x)\} \) are also continuous in \( z, x \), and the integrals are all nonnegative, \( \Psi_w[\Psi_m[\Gamma_w]] \) is also continuous in \( x \) for any nonnegative function \( \Gamma_w \). Similarly, \( \Psi_m[\Psi_w[\Gamma_w]] \) is also bounded and continuous, so that any solution of the fixed point problem in (3.5), if one exists, must be continuous.

Hence, the range of the operators \( \Psi_w \circ \Psi_m \) and \( \Psi_m \circ \Psi_w \) is restricted to a set of bounded continuous functions, so that we can w.l.o.g. restrict the fixed point problem to the space of continuous functions satisfying the bounds derived before. Existence of bounded derivatives up to the \( p \)th order follows by
induction using the product rule and existence of bounded partial derivatives of the functions $U(x,z)$ and $V(z,x)$, see Assumption 2.1.

**Contraction mapping:** We next show that the mapping $(\log \Gamma_w, \log \Gamma_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])$ is a contraction on a Banach space of functions that includes all potential solutions of the fixed point problem (3.5). Specifically, let $C^*$ denote the space of continuous functions on $\mathcal{X} \times \mathcal{Z}$ taking nonnegative values and satisfying (B.2) and (B.1). As shown above, any solution to the fixed point problem - if a solution exists - is an element of $C^* \times C^*$, which is a Banach space.

Consider alternative pairs of functions $(\Gamma_w, \Gamma_m)$ and $(\tilde{\Gamma}_w, \tilde{\Gamma}_m)$. Using the definitions of the operators,

$$
\log \Psi_w[\tilde{\Gamma}_m](x) - \log \Psi_w[\Gamma_m](x) = \log \int \frac{\exp \{U(x,s) + V(s,x)\} m(s)}{1 + \exp \{\log \Gamma_m(s)\}} ds
$$

By the mean-value theorem for real-valued functions of a scalar variable, for every value of $x$, there exists $t(x) \in [0,1]$ such that

$$
\frac{1}{\Psi_w[\Gamma_m](x)} \exp \left\{ \frac{\log \Gamma_m(x,t(x))}{\Gamma_m(x)} \right\} \int \frac{\exp \{U(x,s) + V(s,x)\} \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}}{1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}} \left[ \log \Gamma_m(s) - \log \tilde{\Gamma}_m(s) \right] m(s) ds
$$

Since we are restricting our attention to functions $\Gamma_m(z)$. $\tilde{\Gamma}_m(z)$ satisfying the bounds in equation (B.1), we can bound the ratio

$$
0 \leq \frac{\Gamma_m(z)^{1-t(x)} \tilde{\Gamma}_m(z)^{t(x)}}{1 + \Gamma_m(z)^{1-t(x)} \tilde{\Gamma}_m(z)^{t(x)}} \leq \frac{\exp \{\bar{U} + \bar{V}\}}{1 + \exp \{\bar{U} + \bar{V}\}} =: \lambda
$$

(B.3) for all $z \in \mathcal{Z}$. Since all components of the integrand are nonnegative we can bound the right hand side in absolute value by

$$
\left| \frac{\log \Psi_w[\tilde{\Gamma}_m](x)}{\Psi_w[\Gamma_m](x)} \right| \leq \frac{\lambda}{\Psi_w[\Gamma_m](x)} \int \frac{\exp \{U(x,s) + V(s,x)\}}{1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}} \sup_{z \in \mathcal{Z}} \left| \log \tilde{\Gamma}_m(s) - \log \Gamma_m(s) \right| m(s) ds
$$

$$
= \frac{\lambda}{\Psi_w[\Gamma_m](x)} \left\| \log \tilde{\Gamma}_m - \log \Gamma_m \right\| \int \frac{\exp \{U(x,s) + V(s,x)\}}{1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}} m(s) ds
$$

$$
= \lambda \left\| \log \tilde{\Gamma}_m - \log \Gamma_m \right\|_{\infty}
$$

since the integral in the second to last line is equal to $\Psi_w\left[ \frac{\Gamma_m^{1-t(x)} \tilde{\Gamma}_m^{t(x)}}{\Gamma_m(x)} \right](x)$ by definition of the operator $\Psi_w$. Since this upper bound does not depend on the value of $x$, it follows that

$$
\left\| \log \Psi_w[\tilde{\Gamma}_m] - \log \Psi_w[\Gamma_m] \right\|_{\infty} = \sup_{x \in \mathcal{X}} \left\| \log \Psi_w[\tilde{\Gamma}_m](x) - \log \Psi_w[\Gamma_m](x) \right\|
$$

$$
\leq \lambda \left\| \log \tilde{\Gamma}_m - \log \Gamma_m \right\|_{\infty}
$$

and by a similar argument,

$$
\left\| \log \Psi_m[\tilde{\Gamma}_w] - \log \Psi_m[\Gamma_w] \right\|_{\infty} \leq \lambda \left\| \log \tilde{\Gamma}_w - \log \Gamma_w \right\|_{\infty}
$$
Since by Assumption 2.1 and the expression in equation (B.3), \( \lambda = \frac{\exp(U+V)}{1+\exp(U+V)} < 1 \), the mapping \((\log \Psi_w, \log \Psi_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])\) is indeed a contraction.

**Existence and uniqueness of fixed point**: Since we showed in the first step that the solution \((\Gamma^*_w, \Gamma^*_m)\), if it exists, has to be continuous, we can take the fixed point mapping \((\log \Gamma \Psi_w, \log \Gamma \Psi_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])\) to be its restriction to the space of continuous functions \((C^\ast \times C^\ast, \| \cdot \|_\infty)\) endowed with the supremum norm
\[
\| (\Gamma_w, \Gamma_m) \|_\infty := \max \left\{ \sup_x |\log \Gamma_w(x)|, \sup_z |\log \Gamma_m(z)| \right\}.
\]
Since this space is a complete vector space, and \((\log \Psi_w, \log \Psi_m)\) is a contraction mapping, the conclusion follows directly using Banach’s fixed point theorem. \(\square\)

In the following, denote \(\tilde{U}_{ij} := U(x_i, z_j)\) and \(\tilde{U}_{ik} := U(x_i, z_k)\). Before proving Lemma 3.1, we are going to establish the following Lemma:

**Lemma B.1.** Suppose that Assumptions 2.1, 2.2, and 2.3 hold, and that the random utilities \(U_0, U_1, \ldots, U_{ij}\) are i.i.d. draws from the model in (2.1) where the outside option is given by (2.2). Then as \(J \to \infty\),
\[
\begin{align*}
&P \left( U_{i0} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{ij} \right) - \frac{1}{1 + \frac{1}{J} \sum_{k=1}^{J} \exp(\tilde{U}_{ik})} \to 0, \text{ and} \\
&J \cdot P \left( U_{ij} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{ij} \right) - \frac{\exp(\tilde{U}_{ij})}{1 + \frac{1}{J} \sum_{k=1}^{J} \exp(\tilde{U}_{ik})} \to 0
\end{align*}
\]
for any fixed \(j = 1, 2, \ldots, J\).

**Proof:** For this proof, denote the \(j\) draws for the outside option in (2.2) with random utilities \(U_{ij} = \tilde{U}_{ij} + \sigma \eta_{ij}\) for \(j = J + 1, \ldots, 2J\), where \(\tilde{U}_{ij} = 0\). Using independence, the conditional probability that \(U_{ij} \geq U_{ik}\) for all \(k = 1, \ldots, J\) given \(\eta_{ij}\) is equal to
\[
P \left( U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{ij}, \eta_{ij} \right) = \prod_{k \neq j} G(\eta_{ij} + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik}))
\]
for any \(j = 0, 1, \ldots, J\). By the law of iterated expectations, we the unconditional probability is obtained by integrating over the density of \(\eta\),
\[
P \left( U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{ij} \right) = \int_{-\infty}^{\infty} \left[ \prod_{k \neq j} G(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})) \right] g(s) ds
\]
\[
= \int_{-\infty}^{\infty} \exp \left\{ \sum_{k \neq j} \log G(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})) \right\} g(s) ds
\]
\[
= \int_{-\infty}^{\infty} \exp \left\{ \sum_{k=0}^{2J} \log G(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})) \right\} \frac{g(s)}{G(s)} ds \tag{B.4}
\]
where the last step follows since \(\tilde{U}_{ij} - \tilde{U}_{ij} = 0\). Now we can rewrite the exponent in the last expression as
\[
\sum_{k=0}^{2J} \log G(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})) = \frac{1}{J} \sum_{k=0}^{2J} J \log \left( G(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})) \right)
\]
We now let the sequences \(b_J := G^{-1}(1 - \frac{1}{J}) \to \infty\) and \(a_J = a(b_J) = \sigma^{-1}\) - where \(a(z)\) is the auxiliary function specified in Assumption 2.2. Then, by a change of variables \(s = a_J t + b_J\), we can rewrite the
Furthermore by Lemma 1.3 in Resnick (1987), we have

\[
P \left( U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_1, \ldots, \tilde{U}_i \right) = \int_{-\infty}^{\infty} \exp \left\{ \frac{1}{J} \sum_{k=0}^{2J} J \log G(a_j(t + \tilde{U}_{ij} - \tilde{U}_{ik}) + b_j) \right\} \frac{a_j g(a_j t + b_j)}{G(a_j t + b_j)} dt
\]

**Convergence of the Integrand.** We next show that for \( j \neq 0 \), the integrand converges to a non-degenerate limit as \( J \to \infty \). First consider the exponent

\[
R_J(t) := \frac{1}{J} \sum_{k=0}^{2J} J \log G(a_j(t + \tilde{U}_{ij} - \tilde{U}_{ik}) + b_j)
\]

Since for \( G \to 1, - \log G \approx 1 - G \), we obtain

\[
R_J(t) = -\frac{1}{J} \sum_{k=0}^{2J} J(1 - G(b_j + a_j(t + \tilde{U}_{ij} - \tilde{U}_{ik}))) + o(1)
\]

where the last step follows from the choice of \( a_j \). Since \((1 - G(s))^{-1}\) is \( \Gamma \)-varying with auxiliary function \( a(s) \), and \( b_j \to \infty \),

\[
\frac{1 - G(b_j + a(b_j)(t + \tilde{U}_{ij} - \tilde{U}_{ik}))}{1 - G(b_j)} \to \exp\{ -t - (\tilde{U}_{ij} - \tilde{U}_{ik}) \}
\]

Finally, since \( G(b_j) = 1 - \frac{1}{b_j} \),

\[
J(1 - G(b_j + a(b_j)(t + \tilde{U}_{ij} - \tilde{U}_{ik}))) = \frac{(1 - G(b_j + a(b_j)(t + \tilde{U}_{ij} - \tilde{U}_{ik})))}{1 - G(b_j)} \to \exp\{ -t - (\tilde{U}_{ij} - \tilde{U}_{ik}) \}
\]

Since \( G(a_j t + b_j) \) is also nondecreasing in \( t \), convergence of the integrand is also locally uniform with respect to \( t \) and \((\tilde{U}_{ij} - \tilde{U}_{ik})\) by the arguments in section 0.1 in Resnick (1987). Hence,

\[
R_J(t) = -e^{-t} \frac{1}{J} \sum_{k=0}^{2J} \exp\{ \tilde{U}_{ik} - \tilde{U}_{ij} \} + o(1) \tag{B.5}
\]

where the term \( \frac{1}{J} \sum_{k=0}^{2J} \exp\{ \tilde{U}_{ik} - \tilde{U}_{ij} \} \leq 1 + \exp\{2U\} < \infty \) is uniformly bounded by Assumption 2.1. Next, we turn to the term

\[
r_J(t) = Ja_j g(b_j + a_j t) = Ja(b_j)g(b_j + a_j t)
\]

Since \( a(z) = \frac{1 - G(z)}{g(z)} \), we can write

\[
r_J(t) = Ja(b_j) \frac{1 - G(b_j + a_j t)}{a(b_j + a_j t)}
\]

By the same steps as before,

\[
J(1 - G(b_j + a_j t)) \to e^{-t}
\]

Furthermore by Lemma 1.3 in Resnick (1987), we have

\[
\frac{a(b_j)}{a(b_j + a_j t)} \to 1
\]

so that

\[
r_J(t) \to e^{-t}
\]
Combining this result with (B.5), we get
\[ J \exp \left\{ \frac{1}{J} \sum_{k=0}^{2J} J \log G(a_{ij}(t + \hat{U}_{ij} - \hat{U}_{ik}) + b_{ij}) \right\} a_{ij}(a_{ij}t + b_{ij}) = \exp\{R_J(t)\}r_J(t) \]
\[ = \exp \left\{ -t - e^{-\frac{1}{J} \sum_{k=0}^{2J} \exp\{\hat{U}_{ik} - \hat{U}_{ij}\}} \right\} + o(1) \]
for every \( t \in \mathbb{R} \).

**Convergence of the Integral.** Let \( h_J^*(t) := \exp \left\{ -t - \frac{1}{J} \sum_{k=0}^{2J} \exp\{\hat{U}_{ik} - \hat{U}_{ij}\} \right\} \). Since the function \( h_J(t) := \exp\{R_J(t)\}r_J(t) \) is bounded uniformly in \( J \), and \( |h_J(t) - h_J^*(t)| \to 0 \) pointwise, it follows that
\[ \left| JP \left( U_{ij} \geq U_{ik}, k = 0, \ldots, J | \hat{U}_{i1}, \ldots, \hat{U}_{ij} \right) - \int_{-\infty}^{\infty} h_J^*(t)dt \right| = \left| \int_{-\infty}^{\infty} (h_J(t) - h_J^*(t))dt \right| \to 0 \]
using dominated convergence. From a change in variables \( \psi := -e^{-t} \), we can evaluate the integral
\[ \int_{-\infty}^{\infty} h_J^*(t)dt = \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{J} \sum_{k=0}^{2J} \exp\{\hat{U}_{ik} - \hat{U}_{ij}\} \right\} e^{-t}dt \]
\[ = \left( \frac{1}{J} \sum_{k=0}^{2J} \exp\{\hat{U}_{ik} - \hat{U}_{ij}\} \right)^{-1} \frac{\exp\{\hat{U}_{ij}\}}{\frac{1}{J} \sum_{k=0}^{2J} \exp\{\hat{U}_{ik}\}} \]
\[ = \frac{\exp\{\hat{U}_{ij}\}}{1 + \frac{1}{J} \sum_{k=1}^{2J} \exp\{\hat{U}_{ik}\}} \]
since \( \sum_{j=0}^{2J} \exp\{\hat{U}_{ij}\} = J \). Hence,
\[ \left| JP \left( U_{ij} \geq U_{ik}, k = 0, \ldots, J | \hat{U}_{i1}, \ldots, \hat{U}_{ij} \right) - \frac{\exp\{\hat{U}_{ij}\}}{1 + \frac{1}{J} \sum_{k=1}^{2J} \exp\{\hat{U}_{ik}\}} \right| \to 0 \]
for each \( j = 1, 2, \ldots, J \), as claimed in (B.1). Furthermore, it follows that
\[ P \left( U_{i0} \geq U_{i1}, k = 0, \ldots, J | \hat{U}_{i1}, \ldots, \hat{U}_{ij} \right) = 1 - \sum_{j=1}^{J} P(U_{ij} \geq U_{ik}, k = 0, \ldots, J) \quad (B.6) \]
\[ = \frac{1}{1 + \frac{1}{J} \sum_{j=1}^{2J} \exp\{\hat{U}_{ik}\}} + o(1) \]
which establishes the second assertion. \( \square \)

**B.2. Proof of Lemma 3.1.** For the main conclusion of the lemma, note that since \( z_1, z_2, \ldots \) are a sequence of i.i.d. draws from \( M(z) \), Assumption 2.1 and a law of large numbers can be used to establish \( \frac{1}{T} \sum_{k=1}^{J} \exp\{U(x_i, z_j)\} \to \int \exp\{U(x_i, s)\} m(s)ds \). It follows by the continuous mapping theorem that
\[ \frac{\exp\{U(x_i, z_j)\}}{1 + \frac{1}{J} \sum_{k=1}^{J} \exp\{U(x_i, z_k)\}} \to \frac{\exp\{U(x_i, z_j)\}}{1 + \int \exp\{U(x_i, s)\} m(s)ds} \]
almost surely, so that the conclusion follows from Lemma B.1 and the triangle inequality. \( \square \)

**B.3. Auxiliary Lemmas for the Proof of Theorem 3.2.** In order to prove Theorem 3.2, we start by establishing the main technical steps separately as Lemmata B.2-B.6. The first result concerns the rate at
which the number of available potential spouses increases for each individual in the market. For a given stable matching \( \mu^* \), we let

\[
J_{wi}^* = \sum_{j=1}^{n} \mathbb{1} \{ V_{ji} \geq V_j^*(W_j^*) \} \quad \text{and} \quad J_{mj}^* = \sum_{i=1}^{n} \mathbb{1} \{ U_{ij} \geq U_i^*(M_i^*) \}
\]

denote the number of men available to woman \( i \), and the number of women available to man \( j \), respectively, where \( M_i^* \) and \( W_j^* \) denote woman \( i \)'s and man \( j \)'s opportunity sets under \( \mu^* \), and \( U_i^*(M) := \max_{j \in M} U_{ij} \) and \( V_j^*(W) := \max_{i \in W} V_{ji} \), where by convention, the outside option \( 0 \in W_j^* \) and \( 0 \in M_i^* \). Similarly, we let

\[
L_{wi}^* = \sum_{j=1}^{n} \mathbb{1} \{ U_{ij} \geq U_i^*(M_i^*) \} \quad \text{and} \quad L_{mj}^* = \sum_{i=1}^{n} \mathbb{1} \{ V_{ji} \geq V_j^*(W_j^*) \}
\]

so that \( L_{wi}^* \) is the number of men to whom woman \( i \) is available, and \( L_{mj}^* \) the number of women to whom man \( j \) is available. Lemma B.2 below establishes that in our setup, the number of available potential matches grows at a root-n rate as the size of the market grows.

**Lemma B.2.** Suppose Assumptions 2.1, 2.2, and 2.3 hold, then (a) under any stable matching,

\[
\frac{n^{1/2} \exp\{-\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} \leq J_{wi}^* \leq \frac{n^{1/2} \exp\{\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}}
\]

\[
\frac{n^{1/2} \exp\{-\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}} \leq J_{mj}^* \leq \frac{n^{1/2} \exp\{\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}}
\]

for each \( i = 1, \ldots, n \) and \( j = 1, \ldots, n \) with probability approaching 1 as \( n \to \infty \). (b) Furthermore,

\[
\frac{n^{1/2} \exp\{-\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} \leq L_{wi}^* \leq \frac{n^{1/2} \exp\{\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}}
\]

\[
\frac{n^{1/2} \exp\{-\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}} \leq L_{mj}^* \leq \frac{n^{1/2} \exp\{\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}}
\]

for each \( i = 1, \ldots, n \) and \( j = 1, \ldots, n \) with probability approaching 1 as \( n \to \infty \).

**Proof:** First note that since the sets of available spouses \( W_i^* \) and \( M_i^* \) under the stable matching are endogenous, the taste shifters \( \eta_{ij} \) and \( \zeta_{ji} \) are in general not independent conditional on those choice sets. To circumvent this difficulty, the following argument only relies on lower and upper bounds on \( U_i^* \) and \( V_i^* \) that are implied by the respective utilities of the outside option \( U_{i0}, V_{j0} \), and unconditional independence of taste shocks.

**Rate for Expectation of upper bound** \( J_{mj}^* \). In the following, we denote the set of women that prefers man \( j \) to their outside option by \( \tilde{W}_j \). Since every woman can choose to remain single, we can bound \( J_{mj}^* \) by

\[
J_{mj}^* = \sum_{i=1}^{n_u} \mathbb{1} \{ i \in W_j^* \} = \sum_{i=1}^{n_u} \mathbb{1} \{ U_{ij} \geq U_i^*(M_i^*) \}
\]

\[
\leq \sum_{i=1}^{n_u} \mathbb{1} \{ U_{ij} \geq U_{i0} \} = \sum_{i=1}^{n_u} \mathbb{1} \{ i \in \tilde{W}_j \} =: \tilde{J}_{mj}
\]

By Assumption 2.3 and Lemma B.1,

\[
JP( U_{ij} \geq U_{i0} | x_i, z_j ) \to \frac{\exp\{\bar{U}_{ij}\}}{1 + \frac{1}{j} \exp\{\bar{U}_{ij}\}}
\]
Hence, we can obtain the expectation of the upper bound \( J_{mj}^* \),

\[
E[\bar{J}_{mj}] \geq \frac{1}{J} \sum_{i=1}^{n_w} \frac{\exp(\bar{U}_{ij})}{1 + J \exp(\bar{U}_{ij})} \leq \frac{n_w}{J} \exp\{\bar{U}\}
\]

where \( \bar{U} < \infty \) was given in Assumption 2.1. Since by Assumption 2.3, \( J = [n^{1/2}] \) and the bound on the right-hand side does not depend on \( z_j, x_1, \ldots, x_n \), we have, by the law of iterated expectations that

\[
E[\bar{J}_{mj}] \leq n^{1/2}(\exp(\bar{U} + \gamma_w) + o(1))
\]

where the remainder term \( o(1) \) can be shown to converge uniformly for \( j = 1, 2, \ldots \).

**Rate for Variance of \( \bar{J}_{mj} \).** Let \( p_{ijn} := \frac{\exp(\bar{U}_{ij})}{J + \exp(\bar{U}_{ij})} \) and \( \bar{v}_{jn} := \frac{1}{n} \sum_{i=1}^{n_w} p_{ijn}(1 - p_{ijn}) \). Since by Assumption 2.3, \( \sqrt{n+1} \exp(-\bar{U}) \leq p_{ijn} \leq \frac{\exp(\bar{U})}{\sqrt{n+1} \exp(\bar{U})} \), we have that \((n^{1/2} + 2)^{-1} \exp\{-\bar{U} + \gamma_w\} \leq \bar{v}_{jn} \leq n^{-1/2} \exp\{\bar{U} + \gamma_w\}\). Hence, \( \bar{v}_{jn} \to 0 \) and \( n\bar{v}_{jn} \to \infty \).

Since \( \eta_{a,k}, k = 1, \ldots, J \) are i.i.d. draws from the distribution \( G(\eta) \), we can apply a CLT for independent heterogeneously distributed random variables to the upper bound \( \bar{J}_{mj} \),

\[
\frac{\bar{J}_{mj} - E[\bar{J}_{mj}]}{\sqrt{n\bar{v}_{jn}}} = \frac{1}{\sqrt{n\bar{v}_{jn}}} \sum_{i=1}^{n_w} (\mathbb{1}\{U_{ij} \geq U_{i0}\} - p_{ijn}) \xrightarrow{d} \mathcal{N}(0,1)
\]

where the Lindeberg condition holds since the random variables \( \mathbb{1}\{U_{ij} \geq U_{i0}\} \) are bounded, and \( n\bar{v}_{jn} \to \infty \). Since \( \bar{v}_{jn} \to 0 \) uniformly in \( j = 1, 2, \ldots \), we obtain that

\[
\frac{\bar{J}_{mj} - E[\bar{J}_{mj}]}{\sqrt{n}} \xrightarrow{p} 0
\]

uniformly in \( j = 1, 2, \ldots \).

**Rate for Expectation of lower bound \( J_{wi}^* \).** Next, we denote the set of men \( j \) that prefer woman \( i \) to their outside option or any woman in \( W_j \) by \( M_i^* \). Since by construction, \( W_j \) is a superset of (i.e. contains) \( W_j^* \), \( M_i^* \subset M_i^* \). Hence, we can bound \( J_{wi}^* \) by

\[
J_{wi}^* = \sum_{j=1}^{n_m} \mathbb{1}\{j \in M_i^*\} = \sum_{j=1}^{n_m} \mathbb{1}\{V_{ji} \geq V_j^*(W_j^*)\} \\
\geq \sum_{j=1}^{n_m} \mathbb{1}\{V_{ji} \geq V_j^*(W_j)\} = \sum_{j=1}^{n_m} \mathbb{1}\{j \in M_i^*\} =: J_{wi}^*
\]

Applying Lemma B.1 again, we obtain

\[
JP \left( V_{ji} \geq \max_{k \in W_j} V_{jk} \mid x_i, z_j, W_j \right) \xrightarrow{\text{Law}} \frac{\exp(\bar{V}_{ji})}{1 + J \sum_{k \in W_j} \exp(\bar{V}_{jk})} \geq \frac{J \exp\{-\bar{V}\}}{J + J_{mj} \exp\{V\}}
\]

where \( \bar{V} < \infty \) was defined in Assumption 2.1.

Finally, note that this lower bound is a convex function of \( \bar{J}_{mj} \), so that we can use our previous bound in (B.8) together with Jensen’s Inequality to obtain

\[
JP \left( V_{ji} \geq \max_{k \in W_j} V_{jk} \mid x_i, z_j \right) \geq \frac{J \exp\{-\bar{V}\}}{J + E[\bar{J}_{mj}] \exp\{V\}} \geq \frac{J \exp\{-\bar{V}\}}{J + n^{1/2} \exp\{\bar{U} + \gamma_w + V\}}
\]
which is bounded for all values of $J$ since $J = \lceil \sqrt{n} \rceil$. Hence, by the law of iterated expectations, we can obtain the expectation of the lower bound $J^o_{wi}$,

$$
E[J^o_{wi}] = \sum_{j=1}^{n_m} P \left( V_{ji} \geq \max_{k \in W_j} V_{jk} | x_i, z_j \right) \geq n^{1/2} \left( \exp\{-\bar{V} + \gamma_m\} \right) + o(1)
$$

where the remainder term $o(1)$ can be shown to converge uniformly for $i = 1, 2, \ldots$.

**Rate for Variance $J^o_{wi}$.** Let $p_{jin} := \frac{\exp(V_{ji})}{J \sum_{k \in W_j} \exp(V_{ik})}$ and $\bar{v}_{in} := \frac{1}{n} \sum_{j=1}^{n_m} p_{jin} (1 - p_{jin})$. Using the corresponding bounds derived above and similar steps as for $\bar{v}_{in}$, we obtain $\bar{v}_{in} \rightarrow 0$ and $n \bar{v}_{in} \rightarrow \infty$. Since $\zeta_{j0,k}$, $k = 1, \ldots, J$, and $\zeta_{ji}$, $i = 1, \ldots, n$ are i.i.d. draws from the distribution $G(\eta)$ and independent of $\bar{W}_j$, we can again apply the Lindeberg-Lévy CLT to obtain

$$
\frac{J^o_{wi} - E[J^o_{wi}]}{\sqrt{n} \varepsilon} \rightarrow 0
$$

uniformly in $i = 1, 2, \ldots$.

**Symmetry: Bounds for both sides.** If we reverse the role of the male and female sides of the market, we can repeat the same sequence of steps and obtain a lower bound $J^o_{mj} \leq J^*_{mj}$ and an upper bound $\bar{J}_{wi} \geq J^o_{wi}$ satisfying

$$
\begin{align*}
E[J_{wi}] &\leq n^{1/2} (\exp(\bar{V} + \gamma_m) + o(1)) \\
E[J^o_{mj}] &\geq n^{1/2} \left( \frac{\exp(-\bar{U} + \gamma_w)}{1 + \exp(U + V + \gamma_m)} + o(1) \right)
\end{align*}
$$

where

$$
\frac{\bar{J}_{wi} - E[\bar{J}_{wi}]}{\sqrt{n} \varepsilon} \rightarrow 0 \quad \text{and} \quad \frac{J^o_{mj} - E[J^o_{mj}]}{\sqrt{n} \varepsilon} \rightarrow 0
$$

which concludes the proof of part (a). The proof of part (b) is completely analogous. \qed

We now show that an exogenous change to an arbitrarily chosen availability indicator affects a given individual’s opportunity set with a probability that converges to zero as $n$ grows. In the following we use indices $i$ and $k$ to denote a specific (generic, respectively) woman in the market, and $j$ and $l$ to denote a specific (generic) man. The indicator variable $D^*_i := \mathbb{1} \{ i \in M^*_j \}$ is equal to 1 if man $l$ is available to woman $i$ under the stable matching $\mu^*$, and zero otherwise. Similarly, we let $E^*_{jk} := \mathbb{1} \{ k \in W_j^* \}$ be an indicator variable that is equal to 1 if woman $k$ is available to $j$, and zero otherwise. We can stack the indicator variables whether men $l = 1, \ldots, n_m$ are available to woman $i$ under the stable matching $\mu^*$ to form the vector $D^*_i := (D^*_i, \ldots, D^*_{im})'$, and dummies whether women $k = 1, \ldots, n_w$ are available to man $j$ to form the vector $E^*_j := (E^*_j, \ldots, E^*_{jn})'$. In the following, we also use $D^*_j, E^*_j, D^*_i, E^*_i$ to denote the corresponding vectors of availability indicators under the W- and M-preferred matchings.

The following lemma gives a bound on the probability that changing one arbitrarily chosen availability indicator from one to zero (or vice versa) alters another agent’s opportunity set, where the bound converges to zero at a root-$n$ rate.

**Lemma B.3.** Suppose Assumptions 2.1, 2.2, and 2.3 hold, and let \{ $D^*_{kl}, E^*_{lk} : k = 1, \ldots, n_w, l = 1, \ldots, n_m$ \} be the availability indicators arising from a stable matching. Suppose we change $E^*_{ji}$ exogenously to $\bar{E}^*_{ji} = 1 - E^*_{ji}$ for some woman $i$ and man $j$ and then iterate the deferred acceptance algorithm from that starting point to convergence. Denoting the resulting availability indicators with \{ $\bar{D}^*_{kl}, \bar{E}^*_{lk} : k = 1, \ldots, n_w, l = 1, \ldots, n_m$ \},
for any woman \( \tilde{k} \) and man \( \tilde{l} \neq j \) we have that (a)

\[
P(\tilde{D}_{\tilde{k}} \neq D^*_k | D^*_k, D^*_{ij} = 0) = P(\tilde{E}_{\tilde{l}} \neq E^*_l | E^*_l, D^*_{ij} = 0) = 0
\]

and (b) there exist constants \( \bar{q} < \infty \) and \( 0 < \lambda < 1 \) such that we can bound the conditional probabilities

\[
P(\tilde{D}_{\tilde{k}} \neq D^*_k | D^*_k, D^*_{ij} = 1) \leq \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]

\[
P(\tilde{E}_{\tilde{l}} \neq E^*_l | E^*_l, D^*_{ij} = 1) \leq \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]

The analogous result holds for an exogenous change of an availability indicator \( D_{ij} \) exogenously to \( \tilde{D}_{ij} = 1 - D^*_{ij} \).

**Proof:** In the following, we let \( \tilde{D}^{(s)}_{kl} \) denote the indicator whether man \( l \) is available to woman \( k \) after the \( s \)th iteration, \( \tilde{E}^{(s)}_{kl} \) an indicator whether woman \( k \) is available to man \( l \) after the \( s \)th iteration of the algorithm, and \( \tilde{V}^{(s)}_j := \max_{k: E_{kj} = 1} \{ U_{kl} \} \) and \( \tilde{V}^{(s)}_k := \max_{l: E_{lk} = 1} \{ V_{lk} \} \) woman \( k \) and man \( l \)’s respective indirect utility given their opportunity sets at the \( s \)th stage.

We first show that switching one of \( i \)’s availability indicators and then following the resulting proposals and rejections in the Gale-Shapley algorithm starts a “chain” of subsequent changes, where at each iteration, there is at most one element in each of the two sets of dummies \( \{ \tilde{D}^{(s)}_{kl} \}_{k,l} \) and \( \{ \tilde{E}^{(s)}_{kl} \}_{k,l} \) that will be changed at the \( s \)th stage and has an impact on subsequent rounds. Furthermore, at each iteration, there is a nontrivial probability that the shift in the previous iteration only affects the outside option, in which case the chain will be terminated at that stage.

**Base Case.** First, note that changing \( E_{ji} \) exogenously to \( \tilde{E}^{(1)}_{ji} = 1 - E^*_j \), and leaving the indicators \( \tilde{E}^{(1)}_{ii} = E^*_i \) unchanged for all other men \( l \neq j \), changes only man \( j \)’s indirect utility from \( V^*_j := \max_{k: E_{kj} = 1} \{ V_{lj} \} \) to \( V^{(1)}_j = \max_{k: \tilde{E}^{(1)}_{kj} = 1} \{ V_{lk} \} \). In particular, if \( D_{ij} = 0 \), then \( V_{ji} < V^*_j \), so that \( V^*_j = V^{(1)}_j \). Then any changes to \( E_{ij} \) do not have any subsequent effects on \( j \)’s choices and can therefore be ignored, which establishes part (a) of the lemma. On the other hand, if \( D_{ij} = 1 \), then \( V_{ji} \geq V^*_j \), so that a change from \( E_{ji} = 0 \) to \( \tilde{E}^{(1)}_{ji} = 1 \) increases \( j \)’s indirect utility. Hence if for woman \( k \), \( V_{ki} > V_{kj} \geq V^*_j \), we have that \( D_{kj} = 1 \) and \( \tilde{D}^{(2)}_{kj} = 0 \). Hence it is sufficient to consider shifts in \( E_{ji} \) for men \( j \) such that \( D_{ij} = 1 \).

**Inductive Step.** We now use induction to show that there is at most one such adjustment at each subsequent round \( s = 2, 3, \ldots \) - suppose that after \( s \) iterations of one of the two chains, the availability indicators are given by \( \tilde{D}^{(s)}_{kl} \) and \( \tilde{E}^{(s)}_{kl} \), where \( k = 1, \ldots, n_w \), and \( l = 1, \ldots, n_m \). Under the inductive hypothesis, for the \( s \)th stage there is at most one woman \( k \) such that \( \tilde{E}^{(s)}_k \neq E^{(s-1)}_k \). Furthermore, among all men \( l = 1, \ldots, n_m \) which were available to \( k \) at stage \( s \) (i.e. \( \tilde{D}^{(s)}_{kl} = 1 \)), there was at most one change of an indicator \( \tilde{E}^{(s-1)}_{lk} \) to a new value \( \tilde{E}^{(s)}_{lk} \).

Consider first that the last change from \( \tilde{E}^{(s-1)}_{lk} = 1 \) to \( \tilde{E}^{(s)}_{lk} = 0 \). It follows that \( \tilde{V}^{(s)}_l =: V_{lk'} \) for some \( k' \) such that \( \tilde{E}^{(s-1)}_{lk'} = 1 \), where \( k' \) is unique with probability one. Hence at the \((s + 1)\)st iteration, there is a shift from \( \tilde{D}^{(s)}_{kl} = 0 \) to \( \tilde{D}^{(s-1)}_{kl} = 1 \), i.e. \( l \) becomes available to \( k' \).

Note that man \( l \) also becomes available to any woman \( \tilde{k} \) for whom \( V^*_{\tilde{k}}(s-1) > V_{\tilde{k}l} \geq \tilde{V}^{(s)}_l \). However, by definition of \( k' \), any such \( \tilde{k} \) would not have been available to \( l \), i.e. \( \tilde{E}^{(s)}_{\tilde{k}l} = 0 \). Hence for \( \tilde{k} \), \( U_{\tilde{k}l} < \tilde{V}^{(s)}_l \) so that this change has no effect on subsequent iterations. Note that this includes in particular woman \( k \) who became unavailable to \( j \) at the previous stage. Next, consider a change from \( \tilde{E}^{(s-1)}_{lk} = 0 \) to \( \tilde{E}^{(s)}_{lk} = 1 \), where \( \tilde{D}^{(s)}_{kl} = 1 \) and man \( l \)’s indirect utility in the previous round was \( \tilde{V}^{(s-1)}_l =: V_{lk'} \) for some \( k' \) with \( \tilde{E}^{(s-1)}_{lk'} = 1 \). Since \( \tilde{D}^{(s)}_{kl} = 1 \), it must be true that \( \tilde{V}_{lk} \geq \tilde{V}^{(s-1)}_l \), so that \( l \) may become unavailable to woman
on $k'$, $\tilde{D}^{(s+1)}_{k'} = 0$. On the other hand for any $k'$ such that $V_{lk} = \tilde{V}^{(s)}_{l} > V_{lk} > \tilde{V}^{(s-1)}_{l}$, we must have had $\tilde{E}^{(s)}_{lk} = 0$ by definition of $\tilde{V}^{(s-1)}_{l}$. Hence with probability 1, the change in the $s$th round affects at most one woman $k'$ with $\tilde{E}^{(s)}_{lk'} = 1$, whereas for women $\tilde{k}$ with $\tilde{E}^{(s)}_{lk} = 0$ indirect utility does not depend on whether $l$ is available at round $s + 1$, so that there is no effect on subsequent iterations.

Hence there is at most one indicator corresponding to a woman $k'$ with $\tilde{E}^{(s)}_{lk} = 1$ that changes in the $s$th round. Interchanging the roles of men and women, an analogous argument yields that there is at most one indicator corresponding to a man $l'$ with $\tilde{D}^{(s)}_{kl'} = 1$ that changes in the second part of the $s$th round, confirming the inductive hypothesis.

**Probability of Terminating Events.** Each of the two chains of adjustments can terminate at any given stage $s$ if the change in the previous round only affects the outside option, i.e. if $\tilde{V}^{(s)}_{l} = V_{l0}$ or $\tilde{U}^{(s)}_{k} = U_{k0}$ at $t = s$ or $t = s - 1$. On the other hand if the chain results in a change of $D^{s}_{k_1}, \ldots , D^{s}_{k_{n_m}}$ at a given stage, we ignore any subsequent adjustments and treat such a change as the second terminating event. In the following, we bound the conditional probability for each of these two terminating events given $D^{s}_{k}$ and the chain not having terminated before the $s$th stage.

We first derive a lower bound for the probability that the chain is terminated by the outside option at stage $s$: By the same reasoning as in the proof of Lemma B.2, man $l$’s opportunity set is contained in the set $W_{l}^{s}$, where the taste shifters $\zeta_{ik}$ are jointly independent of $W_{l}^{s}$, and the size of $W_{l}^{s}$ is bounded from above by $n^{1/2} \exp(\bar{U} + \bar{\gamma}_w)$ with probability approaching 1. Hence, by Lemma B.1, we have

$$n^{-1/2} \exp(\bar{V}) \geq \frac{\exp(V(z_{i}, x_{k}))(1 + \exp(V - \bar{U} + \gamma_w))}{n^{1/2}(1 + 2 \exp(V - \bar{U} + \gamma_w))} \geq P(V_{lk} > V_{i}^{*}|x_{k}, z_{i}) \geq \frac{\exp(V(z_{i}, x_{k}))}{n^{1/2}(1 + \exp(U + \bar{V} + \gamma_w))}$$

for any $k, l$ if $n$ is sufficiently large.

This implies that as $n$ grows large, the (unconditional) share of men remaining single is bounded from below by $\frac{1}{1 + \exp(U + \bar{V} + \gamma_w)} =: p_{s}$ with probability approaching 1. By the law of total probability, that bound also holds conditional on $i$’s opportunity set, $(D^{s}_{1}, \ldots , D^{s}_{n_m})$, with probability arbitrarily close to 1 as $n$ grows. Specifically, for the outside option, $P(V_{l0} > V_{i}^{*}|D^{s}_{1}, x_{k}, z_{i}) \geq \frac{1}{1 + \exp(U + \bar{V} + \gamma_w)}$. Furthermore, by Lemma B.2 part (b), the number of women to whom man $l$ is available is bounded by

$$L := n^{1/2} \frac{\exp(-\bar{V} + \gamma_w)}{1 + \exp(U + \bar{V} + \gamma_w)} \leq L_{ml} \leq n^{1/2} \exp(\bar{V} + \gamma_w) =: \bar{L}$$

with probability approaching 1.

In order to construct a lower bound on the conditional probability that man $l$ is unmatched given $\tilde{D}^{(s)}_{kl} = 1$, we can assume the lower bound for $L^{*}_{j}$ if $j$ is unmatched, and the upper bound if $j$ is matched. Then, by Bayes law,

$$P(V_{l0} > V_{i}^{*}|D^{s}_{k}, \tilde{D}^{(s)}_{lk} = 1, x_{k}, z_{i}) \geq \frac{L_{ps}}{L(1 - p_{s}) + L_{ps}} = \frac{n^{-1/2}L}{n^{-1/2}L \exp(U + \bar{V} + \gamma_w) + n^{-1/2}L}$$

which is strictly greater than zero by Assumptions 2.1 and 2.3. Hence, the probability that the shift is not absorbed by the outside option in the $s$th step is less than or equal to

$$1 - P(V_{l0} > V_{i}^{*}|D^{s}_{k}, \tilde{D}^{(s)}_{lk} = 1, x_{k}, z_{i}) \leq \frac{L\exp(U + \bar{V} + \gamma_w)}{L\exp(U + \bar{V} + \gamma_w) + L} =: \lambda < 1$$

where the bound on the right-hand side does not depend on $s$. 
Finally, we construct an upper bound for the probability that the chain leads to a change in the availability indicators \(D_{k1}, \ldots, D_{km} \) at stage \( s \). To this end, we can follow the same reasoning as for the choice probability for the outside option, where we use the lower bound on the size of the opportunity set from Lemma B.2. Applying Lemma B.1, we then have

\[
P(V_{ik} > V^*_i | D^*_k, x_k, z_i) \leq n^{-1/2} \exp(V(z_i, x_k))(1 + \exp(V - U + \gamma_m)) \frac{1}{1 + \exp(V - U + \gamma_m) + \exp(-U - V + \gamma_w)} \leq n^{-1/2} \exp\{V\}
\]

for \( n \) sufficiently large. Hence, the conditional probability that one of the indicators \( D_{ki}, l = 1, \ldots, m \) is switched given that the process is still active at the \( s \)th stage can be bounded by

\[
P(\hat{D}^{(s)}_k \neq D^*_k | D^*_k, \hat{D}^{(s)}_k = 1, z_i, x_k) \leq \frac{n^{-1/2} \exp\{\hat{V}\} \tilde{L}}{n^{-1/2} \exp\{\hat{V}\} \tilde{L} + \tilde{L}} \leq n^{-1/2} \exp\{\hat{V}\} \tilde{L} =: n^{-1/2} \tilde{q}
\]

where \( \tilde{q} < \infty \). Clearly, this upper bound becomes arbitrarily small as \( n \) gets large.

By the law of total probability, the conditional probability that \( \hat{D}^{(s)}_k \neq D^*_k \) can therefore be bounded almost surely by

\[
P(\hat{D}_k \neq D^*_k | D^*_k) \leq \sum_{s=1}^{\infty} \lambda^n n^{-1/2} \tilde{q} \leq \frac{n^{-1/2} \tilde{q}}{1 - \lambda}
\]

which establishes part (b) of the lemma.

Next, we show that the dependence between taste shifters \( \eta_{ij}, \zeta_{ji} \), and opportunity sets becomes small as \( n \) increases. We consider the joint distribution of \( \eta := (\eta_{11}, \ldots, \eta_{n_m})' \), \( \zeta := (\zeta_{11}, \ldots, \zeta_{j_{n_w}})' \) and the availability indicators \( D^W, E^W, D^M, \) and \( E^M \) corresponding to the W- and M-preferred matchings, respectively. We also let \( D^-W_j := (D^W_{i,j-1}, D^W_{i,j+1}, \ldots, D^W_{i,n_m}) \) and \( E^-W_j := (E^W_{i,j-1}, E^W_{i,j+1}, \ldots, E^W_{i,n_m}) \) for the W-preferred matching, and use analogous notation for the M-preferred matching. Then for any vectors of indicator variables \( d = (d_1, \ldots, d_{n_m-1}) \in \{0, 1\}^{n_m-1} \) and \( e = (e_1, \ldots, e_{n_w-1}) \in \{0, 1\}^{n_w-1} \) we denote the conditional c.d.f.s

\[
G^{W}_{D}(\eta|d) := P(\eta \leq \eta | D^W = d), \quad G^{M}_{D}(\eta|d) := P(\eta \leq \eta | D^M = d),
\]

\[
G^{W}_{D,E}(\eta, \zeta|d, e) := P(\eta \leq \eta, \zeta \leq \zeta | D^-W_j = d, E^-W_j = e), \quad G^{M}_{D,E}(\eta, \zeta|d, e) := P(\eta \leq \eta, \zeta \leq \zeta | D^-M_j = d, E^-M_j = e)
\]

with the associated p.d.f.s \( g^{W}_{D}(\eta|d), g^{M}_{D}(\eta|d), \) and \( g^{W}_{D,E}(\eta, \zeta|d, e), \) \( g^{M}_{D,E}(\eta, \zeta|d, e), \) respectively. We also use the analogous notation for the conditional distribution of \( \zeta \) given \( E^W_j \) and \( E^M_j \), respectively.

We can now state the following lemma characterizing the conditional distribution of taste shifters given an agent’s opportunity set.

**Lemma B.4.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then (a) the conditional distributions for \( \eta \) given \( D^W \) and \( D^M \), respectively, satisfy

\[
\lim_n \frac{g^{W}_{D}(\eta|D^W)}{g^{W}_{\eta}(\eta)} - 1 = \lim_n \frac{g^{M}_{D}(\eta|D^M)}{g^{M}_{\eta}(\eta)} - 1 = 0
\]

with probability approaching one as \( n \to \infty \). The analogous results hold for the male side of the market. Furthermore, (b) the conditional distributions for \( (\eta, \zeta) \) given \( D^-W_j, E^-W_j \) (given \( D^-M_j, E^-M_j \), respectively) satisfy

\[
\lim_n \frac{g^{W}_{D,E}(\eta, \zeta|D^-W_j, E^-W_j)}{g^{W}_{\eta, \zeta}(\eta, \zeta)} - 1 = \lim_n \frac{g^{M}_{D,E}(\eta, \zeta|D^-M_j, E^-M_j)}{g^{M}_{\eta, \zeta}(\eta, \zeta)} - 1 = 0
\]
with probability approaching one as \( n \to \infty \). (c) The analogous conclusion holds for any fixed finite subset of men \( M_0 \subset \{1, \ldots, n_m\} \) and women \( W_0 \subset \{1, \ldots, n_w\} \), where the conditioning set excludes the availability indicators between any pair \( k \in W_0 \) and \( l \in M_0 \).

**Proof:** Without loss of generality, let \( \gamma_w = \gamma_m = 0 \). We first prove part (a) for the \( W \)-preferred matching, where we need to establish that the conditional distribution of \( \eta \) given \( D^W_i \) converges to the unconditional distribution at a sufficiently fast rate: Let \( g^{W_i}_{\eta, D}(\cdot) \) denote the p.d.f. of the joint distribution of \( D^W_i \) with \( \eta \). By the definition of conditional densities, we can write

\[
\frac{g^{W_i}_{\eta, D}(\eta | D^W_i)}{g_{\eta}(\eta)} = \frac{g^{W_i}_{\eta, D}(\eta, D^W_i)}{g_{\eta}(\eta)P(D^W_i)} = \frac{P(D^W_i | \eta = \eta)g_{\eta}(\eta)}{P(D^W_i)g_{\eta}(\eta)} = \frac{P(D^W_i | \eta = \eta)}{P(D^W_i)}
\]

where the last step follows since the marginal distributions of \( \eta \) have p.d.f. \( g_{\eta}(\eta) \) by assumption.

The remainder of the proof then derives a common bound on the relative change in the conditional probability of \( D^W_i \) given \( \eta \) that does not depend on \( D^W_i \) and \( \eta_i \), and apply that bound to the event \( D^W_i \) to establish that \( \left| \frac{P(D^W_i | \eta_i = \eta_i)}{P(D^W_i)} - 1 \right| \to 0 \) almost surely. Hence, as a final step it follows from (B.9) that

\[
\left| \frac{\overline{g^{W_i}_{\eta, D}(\eta | D^W_i)}}{g_{\eta}(\eta)} - 1 \right| \to 0.
\]

Specifically, let \( \bar{\eta} = (\eta_1, \ldots, \eta_{n_w})' \), \( \bar{\zeta} = (\zeta_1', \ldots, \zeta_{n_m}')' \), \( \bar{\eta}_{-i} = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{n_w})' \), and define the random variable

\[
I(\eta, d^W) := \mathbbm{1}\{\bar{\eta}_{-i}, \eta, \bar{\zeta} \text{ imply } D^W_i = d^W\}
\]

an indicator whether \( D^W_i \) results from the \( W \)-preferred stable matching given the realizations of taste shifters. We can then write

\[
P(D^W_i = d^W | \eta_i = \eta_i) = \int I(\eta_i, d^W) dG(\bar{\eta}, \bar{\zeta} | \eta_i = \eta_i) = \int I(\eta_i, d^W) dG(\bar{\eta}, \bar{\zeta})
\]

since \( \eta_i \) and \( \bar{\eta}_{-i}, \bar{\zeta} \) are (unconditionally) independent by assumption.

Now for any pair of alternative values \( \eta_1, \eta_2 \), we can then bound

\[
P(D^W_i = d^W | \eta_i = \eta_1) - P(D^W_i = d^W | \eta_i = \eta_2) = \frac{(I(\eta_i, d^W) - I(\eta_i, d^W)) dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D^W_i = d^W | \eta_i = \eta_1)} \leq \frac{I(\eta_i, d^W)(1 - I(\eta_i, d^W)) dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D^W_i = d^W | \eta_i = \eta_1)} \tag{B.10}
\]

and

\[
P(D^W_i = d^W | \eta_i = \eta_1) - P(D^W_i = d^W | \eta_i = \eta_2) = \frac{(I(\eta_i, d^W) - I(\eta_i, d^W)) dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D^W_i = d^W | \eta_i = \eta_2)} \leq \frac{I(\eta_i, d^W)(1 - I(\eta_i, d^W)) dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D^W_i = d^W | \eta_i = \eta_2)} \tag{B.11}
\]

It therefore only remains to be shown that the bounds on the right-hand side of (B.10) and (B.11) both converge to zero as \( n \) grows large. We can then combine those two bounds to conclude that

\[
\left| \frac{P(D^W_i = d^W | \eta_i)}{P(D^W_i = d^W)} - 1 \right| \leq \sup_{\eta_1, \eta_2} \left| \frac{P(D^W_i = d^W | \eta_i = \eta_1)}{P(D^W_i = d^W | \eta_i = \eta_2)} - 1 \right| \to 0,
\]

so that claim (a) of the lemma follows (B.9).

**Deferred Acceptance Algorithm.** We now consider the direct effect of a change \( \eta_i \) from \( \eta_i =: \eta_i = (\eta_{1i}, \ldots, \eta_{n_w})' \) to \( \eta_2 = (\eta_{12}, \ldots, \eta_{2n_m})' \) on her opportunity sets under the two extremal matchings, holding all other agents’
taste shifters fixed: By Theorem 2.12 in Roth and Sotomayor (1990), the W- and M-preferred matchings coincide with the solutions of the Gale-Shapley (deferred acceptance) algorithm with the female (male, respectively) side proposing under the assumptions of this paper (see section 2 in their monograph for a detailed description of the algorithm). It is now easy to verify that the result of the deferred acceptance algorithm only depends on which proposals are eventually made and/or rejected, but not their particular order, which may only change the number of iterations needed for the algorithm to converge. Specifically, if \( i \) makes a proposal to a man who is not available to her under the W-preferred matching, that proposal will be rejected at some stage of the algorithm and does not affect the resulting matching.

Among the men who received a proposal from woman \( i \) under the original W-preferred matching but not after the change to \( i \)'s preferences, there is exactly one man \( j \) who was available under the initial matching, i.e. \( D_{ij}^W = 1 \) and \( \tilde{D}_{ij}^W = 0 \) for all \( l \) such that \( E_{li}^W = 1 \) and \( \tilde{E}_{li}^W = 0 \). Similarly, there is exactly one man \( j \) among those receiving a proposal after the change who is also available to \( i \) under the new matching, i.e. \( \tilde{D}_{ij}^W = 1 \) and \( \tilde{\tilde{D}}_{ij}^W = 0 \) for all \( l \) such that \( \tilde{E}_{li}^W = 1 \) and \( E_{li}^W = 0 \). If man \( l \) is unavailable to \( i \) under both matchings, then by Lemma B.3 part (a), a proposal by \( i \) to \( l \) does not alter the resulting stable matching. Any other men who were initially unavailable may have become available under the new matching only as a consequence of changing \( D_{ij} \) and \( D_{ij}^W \), respectively.

**Conditional and Unconditional Probability of \( D_{ij}^W \).** Hence, in order to verify whether a change of \( \eta_i \) from \( \eta_1 \) to \( \eta_2 \) results in a different opportunity set for woman \( i \) under either of the extremal matchings, it is sufficient to verify whether a proposal by \( i \) to her respective spouses under the W-preferred matching given \( \eta_1 \) and \( \eta_2 \), respectively, has an effect on her opportunity set. If such a change does not alter the indicator variables \( D_{i1}, \ldots, D_{im} \), then the same opportunity set is supported by the W-preferred (M-preferred, respectively) matching given the new realization \( \tilde{\eta}_i \) of woman \( i \)'s taste shocks.

Now denote the availability indicators for the W-preferred matching resulting from replacing \( \eta_i = \eta_1 \) with \( \tilde{\eta}_i = \eta_2 \) by \( \tilde{\tilde{D}}_{ij}^W := (\tilde{\tilde{D}}_{i1}^W, \ldots, \tilde{\tilde{D}}_{im}^W)' \). It then follows from Lemma B.3 part (b), that the conditional probability for \( \tilde{\tilde{D}}_{ij}^W \neq D_{ij}^W \) given \( \eta_k \) and \( \tilde{\eta}_i \) can be bounded by

\[
P(\tilde{\tilde{D}}_{ij}^W \neq D_{ij}^W | \eta_k = \eta_1) \leq \sum_{s=1}^{\infty} \lambda^s n^{-1/2} q \leq 2 n^{-1/2} \frac{q}{1 - \lambda}
\]

It follows that

\[
\frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]

which converges to zero as \( n \to \infty \). Similarly, exchanging the roles of \( \eta_1 \) and \( \eta_2 \), as well as \( D_{ij}^W \) and \( \tilde{\tilde{D}}_{ij}^W \), and repeating these steps we can bound

\[
\frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]

In order to show that these two inequalities imply the desired bound, we have to distinguish two cases: If \( P(D_{ij}^W | \eta_k = \eta_2) \geq P(D_{ij}^W | \eta_k = \eta_1) \), then \( \frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 > 0 \) so that by first inequality,

\[
\left| \frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \right| = \frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]

If on the other hand \( P(D_{ij}^W | \eta_k = \eta_2) \leq P(D_{ij}^W | \eta_k = \eta_1) \), then \( \frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \geq 0 \) so that the second inequality also holds in absolute values. Since in that case we also have

\[
\left| \frac{P(D_{ij}^W | \eta_k = \eta_2)}{P(D_{ij}^W | \eta_k = \eta_1)} - 1 \right| = \left| \frac{P(D_{ij}^W | \eta_k = \eta_1)}{P(D_{ij}^W | \eta_k = \eta_2)} \right| - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}
\]
Hence the upper bound is the same in both cases, so that
\[
\left| \frac{P(D_i^W|\eta_i = \eta_2)}{P(D_i^W|\eta_i = \eta_1)} - 1 \right| \leq \frac{n^{-1/2}q}{1 - \lambda}
\]
which converges to zero. Combining the two bounds with (B.9) yields the conclusion of part (a) for the W-preferred matching. The argument for the M-preferred matching is completely analogous.

For parts (b) and (c), note that the argument in part (a) can be extended directly from one to any finite number of individuals. Specifically if we change the values of \(\eta_i\) and \(\zeta_j\) in an arbitrary manner, we generate four rather than two chains of adjustments, whereas at any iteration, each chain can affect either i's or j's opportunity set. Hence, we can bound the probability of a shift by a multiple of the bound in part (a), \(\frac{4n^{-1/2}q}{1 - \lambda}\), which can in turn be made arbitrarily small by choosing \(n\) large enough. Part (c) can be established in a similar fashion.

In the following, let \(I^M_{wi} = I_{wi}[M^M_i]\) and \(I^W_{wi} = I_{wi}[M^W_i]\) denote the inclusive values for woman \(i\) under the two extremal matchings, so that for any other stable matching, \(I^M_{wi} \leq I_{wi}[M^*_i] \leq I^W_{wi}\). Also, let \(\Gamma^M_w(x)\) and \(\Gamma^W_w(x)\) be the corresponding average inclusive value functions. Similarly, we let \(I^M_{mj} = I_{mj}[W^M_j]\) and \(I^W_{mj} = I_{mj}[W^W_j]\) be the men's inclusive values, and \(\Gamma^M_m(z)\) and \(\Gamma^W_m(z)\) the corresponding average inclusive value functions.

**Lemma B.5.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then, (a) for the M-preferred stable matching,
\[
I^M_{wi} \geq \hat{\Gamma}^M_{wn}(x_i) + o_p(1) \text{ and } I^M_{mj} \leq \hat{\Gamma}^M_{mn}(z_j) + o_p(1)
\]
for all \(i = 1, \ldots, n_w\) and \(j = 1, \ldots, n_m\). Furthermore, (b), if the weight functions \(\omega(x, z) \geq 0\) are bounded and form a VC class in \((x, z)\), then
\[
\sup_{x \in X} \frac{1}{n} \sum_{j=1}^{n_m} \omega(x, z_j)(I^M_{mj} - \hat{\Gamma}^M_m(z_j)) \leq o_p(1) \text{ and } \inf_{z \in Z} \frac{1}{n} \sum_{j=1}^{n_m} \omega(x_i, z)(I^M_{wi} - \hat{\Gamma}^M_w(x_i)) \geq o_p(1).
\]
The analogous conclusions hold for the W-preferred stable matching with the inequalities reversed.

**Proof:** First, note that we can bound conditional choice probabilities given an opportunity set from a pairwise stable matching using the extremal matchings: Specifically, we define
\[
\Lambda^M_w(x, z; M^M) := P(U_{ij} \geq U^*_i(M^M_i)|M^M = M^M_i, x_i = x, z_j = z)
\]
as the conditional choice probability given the realization of the opportunity set \(M^M\) from the male-preferred matching. Also, we let the function \(\Lambda_w(x, z, W)\) be the conditional choice probability for an exogenously fixed opportunity set \(M^M\),
\[
\Lambda_w(x, z; M) := P(U_{ij} \geq U^*_i(M)|x_i = x, z_j = z)
\]
By Lemma B.4, the conditional distribution of taste shifters \(\eta_i\) given \(W^M_i\) is approximated by its marginal distribution as \(n\) grows large. Hence, combining Lemmas B.1 and B.4, there exists a selection of stable matchings \(\mu^*\) such that \(M^*_i = M^M_i\) w.p.a.1, and taste shifters are independent of \(M^*_i\). In particular, we have
\[
J\Lambda^M_w(x_i, z; M^M_i) \leq J\Lambda_w(x_i, z; M^M_i) + o_p(1)
\]
Furthermore, the conditional success probabilities $\Lambda_w(\cdot; I_w)$ and $\Lambda_m(\cdot; I_m)$ are of the order $J^{-1} = n^{-1/2}$, whereas the approximation error in Lemma B.4 is multiplicative. Hence, by Lemma B.4 part (b),

$$\mathbb{E}[J(D_{U_1}^M - \Lambda_m(z_{t1}, x_i; I_{ml1}))(I_{ml1}, I_{ml2}, x_i, z_j) \to 0,$$

and

$$\mathbb{E}[J^2(D_{U_1}^M - \Lambda_m(z_{t1}, x_i; I_{ml1}))(D_{U_1}^M - \Lambda_m(z_{t2}, x_i; I_{ml2}))(I_{ml1}, I_{ml2}, x_i, z_j) \to 0$$

with probability approaching 1, and for all $l_1 = 1, \ldots, n_m$ and $l_2 \neq l_1$. Therefore by the law of iterated expectations, and the conditional variance identity we have that for any two men $l_1 \neq l_2$ the unconditional pairwise covariance

$$J^2\text{Cov}((D_{U_1}^M - \Lambda_m(z_{t1}, x_i; I_{ml1})), (D_{U_1}^M - \Lambda_m(z_{t2}, x_i; I_{ml2}))) \to 0$$

with probability approaching 1. Since by Assumption 2.1 $\exp \{U(x, z)\}$ is bounded by a constant, we have that $\text{Var}(I_{ml}^M - \Gamma_{ml}(x_i)) \to 0$ for each $i = 1, \ldots, n_w$, so that part (a) follows from Chebyshev’s Inequality.

For part (b), it is sufficient to notice that part (a) and boundedness of $\omega(x, z)$ imply joint convergence in probability for any finite grid of values $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$, so that uniform convergence follows from the VC condition on $\omega(x, z)$ following standard arguments. □

Next, we establish uniform convergence of the fixed point mapping $\Psi$ in equation (3.1). We consider uniformity with respect to $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$, where $\mathcal{T}_w$ and $\mathcal{T}_m$ denote the space of bounded continuous real-valued functions on $\mathcal{X}$ and $\mathcal{Z}$, respectively, whose values and first $p$ partial derivatives are bounded by constants larger or equal to those from Theorem 3.1.

Recall that $\hat{\Psi}_w[\Gamma_m](x)$ as defined in (3.1) is the sample average

$$\hat{\Psi}_{wn}[\Gamma_m](x) = \frac{1}{n} \sum_{j=1}^{n_m} \psi_w(z_j, x; \Gamma_m)$$

where

$$\psi_w(z_j, x; \Gamma_m) := \frac{\exp \{U(x, z_j) + V(z_j, x)\}}{1 + \Gamma_m(z_j)}$$

Similarly, we denote

$$\psi_m(x_i, z; \Gamma_w) := \frac{\exp \{U(x_i, z) + V(z, x_i)\}}{1 + \Gamma_w(x_i)}$$

and define the classes of functions $\mathcal{F}_w : \{\psi_w(\cdot, x; \Gamma_m) : x \in \mathcal{X}, \Gamma_m \in \mathcal{T}_m\}$ and $\mathcal{F}_m : \{\psi_m(\cdot, z; \Gamma_w) : z \in \mathcal{Z}, \Gamma_w \in \mathcal{T}_w\}$.

**Lemma B.6.** Suppose Assumption 2.1 holds. Then (i) the classes $\mathcal{F}_w$ and $\mathcal{F}_m$ are Donsker, and (ii) the mapping

$$\left(\hat{\Psi}_w[\Gamma_m](x), \hat{\Psi}_m[\Gamma_w](x)\right) \overset{P}{\to} (\Psi_w[\Gamma_m](x), \Psi_m[\Gamma_w](z))$$

uniformly in $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$ and $(x', z') \in \mathcal{X} \times \mathcal{Z}$ as $n \to \infty$.

**Proof:** The Donsker property follows from fairly standard arguments: By Assumption 2.1, the function $\exp \{U(x, z; \theta) + V(z, x; \theta)\}$ is Lipschitz continuous in each of its arguments. Following Example 19.7 in van der Vaart (1998), $\mathcal{G} := \{\exp \{U(x, z; \theta) + V(z, x; \theta)\} : x \in \mathcal{X}, z \in \mathcal{Z}, \theta \in \Theta\}$ is a Vapnik-Cervonenkis (VC) class, and therefore also Donsker. Since by definition of $\mathcal{T}_w, \mathcal{T}_m$, $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$ have $p$ bounded derivatives, the class $\mathcal{H} = \{\Gamma_w \in \mathcal{T}_w\} \cup \{\Gamma_m \in \mathcal{T}_m\}$ satisfies the conditions for Example 19.9 in van der Vaart (1998), and is also VC. Now note that the transformation $\psi(g, h) := \frac{g}{1+h}$ for $g \in \mathcal{G}$ and $h \in \mathcal{H}$ is continuous.
and bounded on its domain since \( g \) and \( h \) are bounded, and \( h \geq 0 \). It then follows from Example 19.20 in van der Vaart (1998) that the class \( \{ \psi(g, h) := \frac{g}{1 + h} \mid g \in G, h \in H \} \) is also Donsker.

To establish (ii), note that the Donsker property of \( F_w, F_m \) implies that the classes are also Glivenko-Cantelli. Hence, \( \hat{\Psi}_w \) and \( \hat{\Psi}_m \) converge uniformly to their respective population expectations. \( \Box \)

**B.4. Proof of Theorem 3.2:** We now turn to the proof of the main theorem, starting with part (a).

**Fixed-point representation.** By Lemma B.5, we have that for the M-preferred matching, \( I^M_{w,i} \geq \hat{\Gamma}^M_{wn}(x_i) + o_p(1) \) and \( I^M_{mj} \leq \hat{\Gamma}^M_{mn}(z_j) + o_p(1) \) for all \( i = 1, \ldots, n_w \) and \( j = 1, \ldots, n_m \). Note that by construction \( I^M_{mj} \geq 0 \) a.s., and \( \exp \{ U(x, z) + V(z, x) \} \leq \exp \{ U + V \} < \infty \) is bounded by Assumption 2.1, and is a VC class of functions in \( x, z \). Hence we can apply Lemma B.5 part (b) to conclude that

\[
\hat{\Gamma}^M_w(x) = \frac{1}{n} \sum_{j=1}^{n_m} \exp \{ U(x, z_j) + V(z_j, x) \} \quad \frac{1}{1 + \hat{\Gamma}^M_{mj}} \geq \frac{1}{n} \sum_{j=1}^{n_m} \exp \{ U(x, z_j) + V(z_j, x) \} + o_p(1)
\]

where the remainder converges to zero in probability uniformly in \( x \). We obtain similar expressions for \( \hat{\Gamma}^M_m(z) \), \( \hat{\Gamma}^W_w(x) \), and \( \hat{\Gamma}^W_m(z) \). Hence, the inclusive value functions satisfy

\[
\hat{\Gamma}^M_w \geq \hat{\Psi}_w[\hat{\Gamma}^M_m] + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_m \leq \hat{\Psi}_m[\hat{\Gamma}^M_w] + o_p(1)
\]

\[
\hat{\Gamma}^W_w \leq \hat{\Psi}_w[\hat{\Gamma}^W_m] + o_p(1) \quad \text{and} \quad \hat{\Gamma}^W_m \geq \hat{\Psi}_m[\hat{\Gamma}^W_w] + o_p(1)
\]

where inequalities are component-wise, i.e., for \( \hat{\Gamma}^M(x) \) and \( \hat{\Gamma}^W(z) \) evaluated at any value of \( x \in \mathcal{X} \) and \( z \in \mathcal{Z} \), respectively. Noting that \( \hat{\Psi}_w[\Gamma_m] \) and \( \hat{\Psi}_m[\Gamma_w] \) are nonincreasing and Lipschitz continuous in \( \Gamma_m \) and \( \Gamma_w \), respectively, we have

\[
\hat{\Gamma}^M_w \geq \hat{\Psi}_w[\hat{\Gamma}^M_m] + o_p(1) \geq \hat{\Psi}_w[\hat{\Psi}_m[\hat{\Gamma}^M_w]] + o_p(1)
\]

from the first two inequalities. Hence, for any functions \( (\Gamma^*_w, \Gamma^*_m) \) solving the fixed-point problem

\[
\Gamma^*_w = \hat{\Psi}_w[\Gamma^*_m] + o_p(1) \quad \text{and} \quad \Gamma^*_m = \hat{\Psi}_m[\Gamma^*_w] + o_p(1)
\]

with equality, we have

\[
\hat{\Gamma}^M_w \geq \Gamma^*_w + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_m \leq \Gamma^*_m + o_p(1)
\]

and, from the second set of inequalities,

\[
\hat{\Gamma}^W_w \leq \Gamma^*_w + o_p(1) \quad \text{and} \quad \hat{\Gamma}^W_m \geq \Gamma^*_m + o_p(1)
\]

However, since the mapping \( \hat{\Psi} \) is a contraction in logs, the fixed point \( (\Gamma^*_w, \Gamma^*_m) \) is unique up to a term converging to zero in probability. Furthermore, since \( M^M_i \subseteq M^W_i \) and \( W^W_j \subseteq W^W_j \) almost surely, we also have

\[
\hat{\Gamma}^M_w \leq \hat{\Gamma}^W_w \quad \text{and} \quad \hat{\Gamma}^M_m \geq \hat{\Gamma}^W_m
\]

It therefore follows that

\[
\hat{\Gamma}^M_w = \Gamma^*_w + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_m = \Gamma^*_m + o_p(1)
\]

and the same condition also holds for the inclusive values from the W-preferred matching. Note that this argument does not require uniformity with respect to (any random selection from) the full set of stable matchings, but only joint convergence for the two extremal matchings.
This establishes the fixed point representation for \( \hat{\Gamma}_W \) and \( \hat{\Gamma}_M \) in equations (3.1) and (3.2). Similarly, we can also establish the fixed point characterization for the inclusive value function \( \hat{\Gamma}_W \) and \( \hat{\Gamma}_M \) for the male side of the market. Since for any other stable matching, \( \hat{\Gamma}_W \leq \hat{\Gamma}_w^* \leq \hat{\Gamma}_W \) and \( \hat{\Gamma}_M \leq \hat{\Gamma}_m^* \leq \hat{\Gamma}_M \), and furthermore by Theorem 3.1, the solution to the exact fixed-point problem \( \Gamma = \hat{\Psi}[\Gamma] \) is unique with probability 1, it follows that (3.2) is also valid for the inclusive value functions under any other stable matching.

In order to prove part (b), we will proceed by the following steps: we first show existence and smoothness of the solutions to the fixed-point problem in the finite economy (3.2), and then show that the solution to the fixed-point problem of the limiting market in (3.5) is well separated, so that uniform convergence of the mapping \( \log \hat{\Psi} \) to \( \log \Psi \) implies convergence of \( \hat{\Gamma} \) to \( \Gamma^* \).

**Existence and smoothness conditions for \( \hat{\Gamma} \).** First, note that existence and differentiability of \( \hat{\Gamma}_w \) and \( \hat{\Gamma}_m \) solving the fixed point problem in (3.2) follows from Theorem 3.1: Since the conditions of the theorem do not make any assumptions on the distribution of \( x_i \) and \( z_j \), it applies to the case in which \( w(x) \) and \( m(z) \) are the p.m.f.s corresponding to the empirical distributions of \( x_i \) and \( z_j \), respectively. Hence, Assumption 2.1 and Theorem 3.1 imply uniqueness and differentiability to \( p \)th order with uniformly bounded partial derivatives conditional on any realization of the empirical distribution. Since the bounds on the contraction constant \( \lambda \) and on partial derivatives of \( \hat{\Gamma}_w \), \( \hat{\Gamma}_m \) do not depend on the marginal distributions, they also hold almost surely with respect to realizations of the empirical distribution.

**Local Uniqueness.** Next, we verify that for all \( \delta > 0 \) we can find \( \eta > 0 \) such that for any pair \( \Gamma, \hat{\Gamma} \) with \( \| \log \hat{\Gamma} - \log \Gamma \|_\infty > \delta \) we have \( \| (\log \hat{\Gamma} - \log \Psi[\hat{\Gamma}]) - (\log \Gamma - \log \Psi[\Gamma]) \|_\infty > \eta \): First, note that by Theorem 3.1, the mapping \( (\log \Gamma) \mapsto (\log \Psi[\Gamma]) \) is a contraction with constant \( \lambda := \frac{\exp(\bar{U} + \bar{V} + \gamma^*)}{1 + \exp(\bar{U} + \bar{V} + \gamma^*)} < 1 \), where we let \( \gamma^* := \max\{\gamma_w, \gamma_m\} \). Then, using the triangle inequality, we can bound

\[
\| (\log \hat{\Gamma} - \log \Psi[\hat{\Gamma}]) - (\log \Gamma - \log \Psi[\Gamma]) \|_\infty \geq \| \log \hat{\Gamma} - \log \Gamma \|_\infty - \| \log \Psi[\hat{\Gamma}] - \log \Psi[\Gamma] \|_\infty \\
\geq \| \log \hat{\Gamma} - \log \Gamma \|_\infty - \lambda \| \log \Gamma - \log \Gamma \|_\infty \\
> (1 - \lambda)\delta > 0
\]

so that we can choose \( \eta = \eta(\delta) := (1 - \lambda)\delta \).

**Convergence of \( \hat{\Gamma} - \Gamma^* \).** Finally, Lemma B.6 implies that the fixed point mapping \( \hat{\Psi} \) converges to \( \Psi_0 \) uniformly in \( x, z \) and \( \Gamma_w \in \mathcal{T}_w \), and \( \Gamma_m \in \mathcal{T}_m \). Since \( \hat{\Psi} > 0 \) is bounded away from zero almost surely, it follows that \( | \log \hat{\Psi} - \log \Psi_0 | \) converges to zero in outer probability and uniformly in \( x, z \) and \( \Gamma_w \in \mathcal{T}_w \), and \( \Gamma_m \in \mathcal{T}_m \) as well. Hence for any \( \varepsilon > 0 \) and \( n \) large enough, we have

\[
P\left( \sup_{\Gamma \in \mathcal{T}} \| \log \hat{\Psi}[\Gamma] - \log \Psi_0[\Gamma] \|_\infty > \frac{\eta}{2} \right) \leq 1 - \varepsilon
\]

It follows from the choice of \( \eta \) above that

\[
P\left( \| \log \hat{\Gamma} - \log \Gamma^* \|_\infty > \delta \right) \leq 1 - \varepsilon
\]

so that convergence of \( \hat{\Gamma} \) to \( \Gamma^* \) in probability under the sup norm follows from the continuous mapping theorem. \( \square \)
B.5. **Proof of Corollary 3.1.** As shown in section 2, the event that woman $i$ and man $j$ are matched under a stable matching requires that woman $i$ prefers $j$ over any man $l$ in her opportunity set $M_i^*$ given that matching, and that man $j$ prefers $i$ over any woman $k$ in his opportunity set $W_j^*$. Now, by Lemmata B.1 and B.4 part (a), the conditional probability that $i$ prefers $j$ over any $l \in M_i^*$ given her inclusive value satisfies
\[
 JP(U_{ij} \geq U_{ij}(M_i^*)|I_{wi}, x_i, z_j) = J \Lambda_w(x_i, z_j; I_{wi}) + o(1)
\]
with probability approaching 1, where $\Lambda_w(\cdot)$ is as defined in section 2.3. Also, by Theorem 3.2 (b), the inclusive values $I_{wi}$ and $I_{mj}$ converge in probability to $\Gamma_w(x_i)$ and $\Gamma_m(z_j)$, respectively, so that by the continuous mapping theorem,
\[
 J \Lambda_w(x_i, z_j; I_{wi}) = J \Lambda_w(x_i, z_j; \Gamma_w(x_i)) + o_p(1).
\]
Similarly, the conditional probability that man $j$ chooses $i$ over every $k \in W_j^*$ converges according to
\[
 JP(V_{ji} \geq V_{ji}(W_j^*)|I_{mj}, x_i, z_j) = J \Lambda_m(z_j, x_i; \Gamma_m(z_j)) + o_p(1).
\]
Finally, by Lemma B.4 part (b) and Assumption 2.3, the joint probability of the two events converges to the product of the marginals,
\[
 nP(U_{ij} \geq U_{ij}(M_i^*), V_{ji} \geq V_{ji}(W_j^*)|I_{wi}, I_{mj}, x_i, z_j) = J^2 \Lambda_w(x_i, z_j; \Gamma_w(x_i)) \Lambda_m(z_j, x_i; \Gamma_m(z_j)),
\]
so that the conclusion of this corollary follows from a LLN using Lemma B.4, part (c), together with Assumptions 2.1 and 2.3, via an argument analogous to the proof of Lemma B.5.

\[\square\]

**REFERENCES**


